ME 115(a): Solution to Homework #2
Winter 2016

Problem 1: (5 points). You were to find the axis and angle of rotation for the following rotation matrix:

$$\begin{bmatrix} 0.882772 & -0.416266 & 0.217798 \\ 0.44976 & 0.882772 & -0.135756 \\ -0.135756 & 0.217798 & 0.966506 \end{bmatrix}$$

Let the matrix entries be denoted a_{ij}, where $i = 1,2,3$ denotes the row, and $j = 1,2,3$ denotes the column. From class notes or the text:

$$\cos(\phi) = \frac{a_{11} + a_{22} + a_{33} - 1}{2} = \frac{0.882772 + 0.882772 + 0.966506 - 1.0}{2} = 0.866025$$

where ϕ is the angle of rotation. Thus, $\phi = \cos^{-1}(0.866025) = 30.000^o$, and as a result $\sin(\phi) = \sin(30^o) = 0.5$. Consequently, $2\sin\phi = 1.0$. Substituting in the appropriate quantities yields:

$$\omega_x = \frac{a_{32} - a_{23}}{2\sin\phi} = 0.217798 + 0.135756 = 0.353554 \quad (1)$$

$$\omega_y = \frac{a_{13} - a_{31}}{2\sin\phi} = 0.217798 + 0.135756 = 0.353554 \quad (2)$$

$$\omega_z = \frac{a_{21} - a_{12}}{2\sin\phi} = 0.44976 + 0.416266 = 0.866026 \quad (3)$$

Note that $\omega_x^2 + \omega_y^2 + \omega_z^2 = 1.0000$.

Problem 2: (10 points) Can every orthogonal matrix be represented by the exponential of a real matrix?

You should have either remembered or derived the fact that $det(e^C) = e^{tr(C)}$, where $tr(C)$ is the trace of determinant C. Note that if $tr(C)$ is real, than $e^{tr(C)}$ is always a positive number and therefore orthogonal matrices with determinant -1 can not be represented as a matrix exponential. This arises from the fact that the Orthogonal Group is a disconnected group. That is, the matrices with +1 determinant are all connected to each other, and similarly the ones with -1 determinant form a continuous subset of $\mathcal{O}(3)$. But, the two subsets are disjoint.

Let's consider this solution in a bit more detail. Note, that if $tr(C) = \frac{\pi}{2}i$ (where $i^2 = -1$), then $det(e^C) = -1$. However, this can not be true if C is real. Recall that the trace of a matrix is equal to the sum of its eigenvalues. Let C be a $n \times n$ matrix. If n is even, then all of the eigenvalues of C must be complex cojugates and or an even number of real roots. Thus, the sum of the eigenvalues must be real. Similarly, if n is odd, the eigenvalues will either be complex conjugates and/or an odd number of real eigenvalues. Thus, the sum of the eigenvalues must also be real number. Thus, if C is a real matrix, then e^C can not represent orthogonal matrices with determinant -1.

Problem 3: (15 points). Do Problems 4(a,b,c) in Chapter 2 of MLS.

Part (a): Let's assume that the statement in part (b) of the problem is true. Let $\vec{w}$ be a 3×1 vector and let $\vec{v}$ be any 3×1 vector. Then:

$$\begin{aligned}
(R\hat{w}R^T)\vec{v} &= R\hat{w}(R^T\vec{v}) \\
&= R(\vec{w} \times (R^T\vec{v})) \\
&= (R\vec{w}) \times (RR^T\vec{v}) \\
&= (R\vec{w}) \times \vec{v} \\
&= \widehat{(R\vec{w})}\vec{v}
\end{aligned}$$

Since this must be true for any vector $\vec{v}$, then $R\hat{w}R^T = (R\vec{w})\hat{}$.

Part (b): We can now assume that part (a) holds.

$$\begin{aligned}
(R\vec{v}) \times (R\vec{w}) &= \widehat{(R\vec{v})}(R\vec{w}) \\
&= (R\hat{v}R^T)(R\vec{w}) \\
&= R\hat{v}R^TR\vec{w} \\
&= R(\hat{v}\vec{w}) \\
&= R(\vec{v} \times \vec{w})
\end{aligned}$$

Part (c): to show that $so(3)$ is a vector space, note that all elements in $so(3)$, which consists of 3×3 skew symmetric matrices, can be written as a linear combination of the following 3 elements of $so(3)$ (which therefore form a basis for $so(3)$):

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{4}$$

Moreover, the linearity requirements on vectors can be seen as follows, for $\alpha, \beta \in \mathbb{R}$, $\vec{v}, \vec{\omega} \in \mathbb{R}^3$, and $\hat{v}, \hat{\omega} \in so(3)$:

- $\alpha\hat{v} \in so(3)$.
- $(\alpha\vec{v} + \beta\vec{\omega})\hat{} = (\alpha\hat{v} + \beta\hat{\omega}) \in so(3)$.

Problem 4: (5 points). Do Problem 5(c) of chapter 2 in the MLS text.

There are two ways to solve this problem. The simplest way is to use the result of part 5(b) quoted in the text:

$$R = \frac{1}{1+||a||^2}\begin{bmatrix} 1+a_1^2-a_2^2-a_3^2 & 2(a_1a_2-a_3) & 2(a_1a_3+a_2) \\ 2(a_1a_2+a_3) & 1-a_1^2+a_2^2-a_3^2 & 2(a_2a_3-a_1) \\ 2(a_1a_3-a_2) & 2(a_2a_3+a_1) & 1-a_1^2-a_2^2+a_3^2 \end{bmatrix} \tag{5}$$

where $||a||^2$ is shorthand notation for $||a||^2 = a_1^2 + a_2^2 + a_3^2$. Noting that

$$trace(R) = \frac{3-||a||^2}{1+||a||^2} \Rightarrow ||a||^2 = \frac{3-trace(R)}{1+trace(R)} = \frac{3-r_{11}-r_{22}-r_{33}}{1+r_{11}+r_{22}+r_{33}}$$

so that an expression for $||a||^2$ is known, simple algebraic manipulation of the off-diagonal term of R yield

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1+||a||^2}{4}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

If you didn't use the results of 5(b) in the text, then you would have started with Cayley's formula $R = (I-\hat{a})^{-1}(I+\hat{a})$ and derived Equation (5).

Problem 5: (10 points). Do Problem 8(b,c) in Chapter 2 of the MLS text.

Solution to 8(b):

$$\begin{aligned} e^{g\Lambda g^{-1}} &= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda g^{-1})^2 + \frac{1}{3!}(g\Lambda g^{-1})^3 + \cdots \\ &= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda^2 g^{-1}) + \frac{1}{3!}(g\Lambda^3 g^{-1}) + \cdots \\ &= g(I + \frac{1}{1!}\Lambda + \frac{1}{2!}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \cdots)g^{-1} \\ &= ge^{\Lambda}g^{-1} \end{aligned}$$

Solution to 8(c):

Problem 6: (5 points). Problem 10(b) in Chapter 2 of the MLS text, but without the issue of surjectivity.

Note that

$$\hat{\omega} = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} = wJ \quad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then:

$$\hat{\omega}^2 = w^2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -w^2 I; \quad \hat{\omega}^3 = -w^3 J$$

Hence the exponetial of $\hat{\omega}$ can be computed as:

$$\begin{aligned}
\exp\left(\theta\hat{\omega}\right) &= \left(I + \frac{\theta}{1!}\hat{\omega} + + \frac{\theta^2}{2!}\hat{\omega}^2 + \cdots\right) \\
&= \left(I + \frac{w\theta}{1!}J - \frac{w^2\theta^2}{2!}I - \frac{w^3\theta^3}{3!}J + \cdots\right) \\
&= \left(1 - \frac{w^2\theta^2}{2!} + \cdots\right) I + \left(\frac{w\theta}{1!} - \frac{w^3\theta^3}{3!} + \cdots\right) J \\
&= \begin{bmatrix} \cos(w\theta) & -\sin(w\theta) \\ \sin(w\theta) & \cos(w\theta) \end{bmatrix}
\end{aligned}$$

While you weren't asked to consider this part of the problem, note that the exponential map from $so(2)$ to $SO(3)$ can not be surjective, as every point in $SO(3)$ can not be covered by every point in $so(2)$. This map is not injective since $\exp(\theta\hat{\omega}) = \exp((\theta + 2\pi)\hat{\omega})$.