Problem 1: (5 points). You were to find the axis and angle of rotation for the following rotation matrix:

\[
\begin{bmatrix}
0.882772 & -0.416266 & 0.217798 \\
0.44976 & 0.882772 & -0.135756 \\
-0.135756 & 0.217798 & 0.966506
\end{bmatrix}
\]

Let the matrix entries be denoted \( a_{ij} \), where \( i = 1, 2, 3 \) denotes the row, and \( j = 1, 2, 3 \) denotes the column. From class notes or the text:

\[
\cos(\phi) = \frac{a_{11} + a_{22} + a_{33} - 1}{2} = \frac{0.882772 + 0.882772 + 0.966506 - 1.0}{2} = 0.866025
\]

where \( \phi \) is the angle of rotation. Thus, \( \phi = \cos^{-1}(0.866025) = 30.000^\circ \), and as a result \( \sin(\phi) = \sin(30^\circ) = 0.5 \). Consequently, \( 2\sin\phi = 1.0 \). Substituting in the appropriate quantities yields:

\[
\omega_x = \frac{a_{32} - a_{23}}{2\sin\phi} = 0.217798 + 0.135756 = 0.353554 \tag{1}
\]

\[
\omega_y = \frac{a_{13} - a_{31}}{2\sin\phi} = 0.217798 + 0.135756 = 0.353554 \tag{2}
\]

\[
\omega_z = \frac{a_{21} - a_{12}}{2\sin\phi} = 0.44976 + 0.416266 = 0.866026 \tag{3}
\]

Note that \( \omega_x^2 + \omega_y^2 + \omega_z^2 = 1.0000 \).

Problem 2: (10 points) Can every orthogonal matrix be represented by the exponential of a real matrix?

You should have either remembered or derived the fact that \( \det(e^C) = e^{tr(C)} \), where \( tr(C) \) is the trace of determinant \( C \). Note that if \( tr(C) \) is real, then \( e^{tr(C)} \) is always a positive number and therefore orthogonal matrices with determinant -1 cannot be represented as a matrix exponential. This arises from the fact that the Orthogonal Group is a disconnected group. That is, the matrices with +1 determinant are all connected to each other, and similarly the ones with -1 determinant form a continuous subset of \( \mathcal{O}(3) \). But, the two subsets are disjoint.

Let’s consider this solution in a bit more detail. Note, that if \( tr(C) = \frac{\pi}{2}i \) (where \( i^2 = -1 \)), then \( \det(e^C) = -1 \). However, this can not be true if \( C \) is real. Recall that the trace of a matrix is equal to the sum of its eigenvalues. Let \( C \) be a \( n \times n \) matrix. If \( n \) is even, then all of the eigenvalues of \( C \) must be complex conjugates and/or an even number of real roots. Thus, the sum of the eigenvalues must be real. Similarly, if \( n \) is odd, the eigenvalues will either be complex conjugates and/or an odd number of real eigenvalues. Thus, the sum
of the eigenvalues must also be real number. Thus, if \( C \) is a real matrix, then \( e^C \) can not represent orthogonal matrices with determinant -1.

**Problem 3:** (15 points). Do Problems 4(a,b,c) in Chapter 2 of MLS.

**Part (a):** Let’s assume that the statement in part (b) of the problem is true. Let \( \vec{w} \) be a \( 3 \times 1 \) vector and let \( \vec{v} \) be any \( 3 \times 1 \) vector. Then:

\[
(R\vec{w}R^T)\vec{v} = R\vec{w}(R^T\vec{v}) \\
= R(\vec{w} \times (R^T\vec{v})) \\
= (R\vec{w}) \times (RR^T\vec{v}) \\
= (R\vec{w}) \times \vec{v} \\
= (\overrightarrow{R\vec{w}})\vec{v}
\]

Since this must be true for any vector \( \vec{v} \), then \( R\vec{w}R^T = (\overrightarrow{R\vec{w}}) \).

**Part (b):** We can now assume that part (a) holds.

\[
(R\vec{v}) \times (R\vec{w}) = (\overrightarrow{R\vec{v}})(R\vec{w}) \\
= (R\vec{v}R^T)(R\vec{w}) \\
= R\vec{v}R^TR\vec{w} \\
= R(\vec{v} \times \vec{w})
\]

**Part (c):** to show that \( so(3) \) is a vector space, note that all elements in \( so(3) \), which consists of \( 3 \times 3 \) skew symmetric matrices, can be written as a linear combination of the following 3 elements of \( so(3) \) (which therefore form a basis for \( so(3) \)):

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Moreover, the linearity requirements on vectors can be seen as follows, for \( \alpha, \beta \in \mathbb{R}, \vec{v}, \vec{w} \in \mathbb{R}^3 \), and \( \hat{v}, \hat{w} \in so(3) \):

- \( \alpha \hat{v} \in so(3) \).
- \( (\alpha \vec{v} + \beta \vec{w}) = (\alpha \hat{v} + \beta \hat{w}) \in so(3) \).

**Problem 4:** (5 points). Do Problem 5(c) of chapter 2 in the MLS text.

There are two ways to solve this problem. The simplest way is to use the result of part 5(b) quoted in the text:
\[ R = \frac{1}{1 + ||a||^2} \begin{bmatrix} 1 + a_1^2 - a_2^2 - a_3^2 & 2(a_1a_2 - a_3) & 2(a_1a_3 + a_2) \\ 2(a_1a_2 + a_3) & 1 - a_1^2 + a_2^2 - a_3^2 & 2(a_2a_3 - a_1) \\ 2(a_1a_3 - a_2) & 2(a_2a_3 + a_1) & 1 - a_1^2 - a_2^2 + a_3^2 \end{bmatrix} \]  

where \( ||a||^2 \) is shorthand notation for \( ||a||^2 = a_1^2 + a_2^2 + a_3^2 \). Noting that 

\[ \text{trace}(R) = 3 - ||a||^2 \Rightarrow ||a||^2 = \frac{3 - \text{trace}(R)}{1 + \text{trace}(R)} = \frac{3 - r_{11} - r_{22} - r_{33}}{1 + r_{11} + r_{22} + r_{33}} \]

so that an expression for \( ||a||^2 \) is known, simple algebraic manipulation of the off-diagonal term of \( R \) yield

\[
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 1 + ||a||^2 \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} / 4
\]

If you didn’t use the results of 5(b) in the text, then you would have started with Cayley’s formula \( R = (I - \hat{a})^{-1}(I + \hat{a}) \) and derived Equation (5).

**Problem 5:** (10 points). Do Problem 8(b,c) in Chapter 2 of the MLS text.

**Solution to 8(b):**

\[
e^{g\Lambda g^{-1}} = I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda g^{-1})^2 + \frac{1}{3!}(g\Lambda g^{-1})^3 + \cdots
\]

\[
= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda^2 g^{-1}) + \frac{1}{3!}(g\Lambda^3 g^{-1}) + \cdots
\]

\[
= g(I + \frac{1}{1!}\Lambda + \frac{1}{2!}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \cdots)g^{-1}
\]

\[
= ge^\Lambda g^{-1}
\]

**Solution to 8(c):**

**Problem 6:** (5 points). Problem 10(b) in Chapter 2 of the MLS text, but without the issue of surjectivity.

Note that

\[
\hat{\omega} = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} = wJ \quad \text{where} \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]

Then:

\[
\hat{\omega}^2 = w^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -w^2I; \quad \hat{\omega}^3 = -w^3J
\]
Hence the exponential of $\mathbf{\hat{\omega}}$ can be computed as:

\[
\exp (\theta \mathbf{\hat{\omega}}) = \left( I + \frac{\theta}{1!} \mathbf{\hat{\omega}} + \frac{\theta^2}{2!} \mathbf{\hat{\omega}}^2 + \cdots \right) \\
= \left( I + \frac{w\theta}{1!} J - \frac{w^2\theta^2}{2!} I - \frac{w^3\theta^3}{3!} J + \cdots \right) \\
= \left( 1 - \frac{w^2\theta^2}{2!} + \cdots \right) I + \left( \frac{w\theta}{1!} - \frac{w^3\theta^3}{3!} + \cdots \right) J \\
= \begin{bmatrix} \cos(w\theta) & -\sin(w\theta) \\ \sin(w\theta) & \cos(w\theta) \end{bmatrix}
\]

While you weren’t asked to consider this part of the problem, note that the exponential map from $so(2)$ to $SO(3)$ cannot be surjective, as every point in $SO(3)$ cannot be covered by every point in $so(2)$. This map is not injective since $\exp(\theta \mathbf{\hat{\omega}}) = \exp((\theta + 2\pi) \mathbf{\hat{\omega}})$. 