ME 115(a): Solution to Homework #3
(Winter 2016)

Problem #1: (20 points, Problem 6(a,b,d,e) in Chapter 2 of MLS).

Part (a): Let Q and P be unit quaternions—i.e., $QQ^* = PP^* = 1$. The set of unit quaternions is a group if you can show that: (i) multiplication is associative; (ii) the product of group elements yields a group element; (iii) the set contains an identity element; (iv) every group element has an inverse element, and the inverse is in the group.

(i) It is easy to show that quaternion multiplication is associative, since multiplication of all of the quaternian basis elements for quaternians is an associative process.

(ii) The product of unit quaternions, QP, is a unit quaternion: $QP(QP)^* = QPP^*Q* = QQ^* = 1$.

(iii) The identity quaternion is: $e = (1, 0, 0, 0) = 1 + 0 \cdot i + 0 \cdot j + 0 \cdot k$.

(iv) The inverse of any unit quaternion Q is Q^*, which is also a unit quaternion (since $Q^*(Q^*)^* = Q^*Q = (QQ^*)^* = 1^* = 1$).

Part (b): If a unit quaternion, q, has real part q_R and pure part q_P, and $\vec{x} = [x_1\ x_2\ x_3]^T$ is represented as a pure quaternion $\tilde{x} = (0, x_1, x_2, x_3) = 0 + \vec{x}$, then:

$$\begin{aligned} \tilde{x}q^{-1} &= \vec{x} \cdot q_P \qquad (\leftarrow \text{real part}) \\ &+ q_R\vec{x} - (\vec{x} \times q_P \qquad (\leftarrow \text{pure part}) \end{aligned}$$

Similarly, the product $q\tilde{x}q^{-1}$ is:

$$\begin{aligned} q\tilde{x}q^{-1} &= q_R(\vec{x} \cdot q_P) - q_P \cdot (q_R\vec{x} - \vec{x} \times q_P) \qquad (\leftarrow \text{real part}) \\ &+ q_R(q_R\vec{x} - \vec{x} \times q_P) + (\vec{x} \cdot q_P)q_P + q_P \times (q_R\vec{x} - \vec{x} \times q_P) \qquad (\leftarrow \text{pure part}) \end{aligned}$$

The real part of $q\tilde{x}q^{-1}$ is:

$$(\vec{x} \cdot q_P)q_R - q_P \cdot [q_R\vec{x} - (\vec{x} \times q_P)] = q_R(\vec{x} \cdot q_P) - q_R(\vec{x} \cdot q_P) + q_P \cdot (\vec{x} \times q_P) = 0$$

Thus $q\tilde{x}q^{-1}$ is a pure quaternion when $\tilde{x}$ is.

The vector part of $q\tilde{x}q^{-1}$ is:

$$\begin{aligned} q_R(q_R\vec{x} - \vec{x} \times q_P) &+ (\vec{x} \cdot q_P)q_P + q_P \times (q_R\vec{x} - \vec{x} \times q_P) \\ &= {q_R}^2\vec{x} - q_R(\vec{x} \times q_P) + (\vec{x} \cdot q_P)q_P + q_R(q_P \times \vec{x}) - q_P \times (\vec{x} \times q_P) \\ &= q_R^2\vec{x} - 2q_R(\vec{x} \times q_P) + (\vec{x} \cdot q_P)q_P - [(q_P \cdot q_P)\vec{x} - (\vec{x} \cdot q_P)q_P] \\ &= [q_R^2 - (q_P \cdot q_P)]\vec{x} + 2[(\vec{x} \cdot q_P)q_P + q_R(q_P \times \vec{x})] \end{aligned}$$

where we have used the triple cross product identity: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Part (d):

(i) If $A_1, A_2 \in SO(3)$, then each of the 9 elements in the product matrix $A_1\ A_2$ requires 3 multiplications and 2 additions. Hence, the product $A_1\ A_2$ requires a total of 27 multiplications and 18 additions.

(ii) Let q_1 and q_2 be quaternions, with respective real and vector parts q_{1R}, q_{2R} and $\vec{q}_{1P}$, $\vec{q}_{2P}$. The real part of the quaternion product, $q_{1R}q_{2R} - \vec{q}_{1P} \cdot \vec{q}_{2P}$, requires 4 multiplications and 3 additions (where the subtraction is counted as an addition). The pure part, $\vec{q}_{3P} = q_{1R}\vec{q}_{2P} + q_{2R}\vec{q}_{1P} + \vec{q}_{1P} \times \vec{q}_{2P}$, can be evaluated in 12 multiplications and 9 additions. Thus, the quaternion product requires a total of 16 multiplications and 12 additions. It is therefore more efficient than the equivalent matrix multiplication.

(iii) The rotation of a vector by multiplication of a 3×3 rotation matrix times a 3×1 vector requires only 9 multiplications and 6 additions.

(iv) The number of multiplications and additions for the equivalent quaternion operation will depend upon the form which one uses for the quaternion vector rotation. Using the identity $1 = q_R^2 + q_P \cdot q_P$, it is possible to show that the vector part of $q\tilde{x}q^{-1}$ in part (b) above can be rearranged to the form:

$$\vec{x} + 2[q_P \times (q_P \times \vec{x}) + q_R(q_P \times \vec{x})]$$

Since $q_P \times \vec{x}$ need only be evaluated once, this takes only 18 multiplications and 12 additions. However, no matter what form one tries, the quaternion approach will always take more operations than the matrix/vector approach for vector rotation.

Part (e): For constant rotation about a fixed unit length axis $\vec{\omega}$, the orientation of a rigid body is described by $R(t) = \exp(\hat{\omega}t)$, where t is time and $\hat{\omega} = (\vec{\omega})^\vee$. The equivalent unit quaternian is

$$q = \left(\cos\left(\frac{t}{2}\right), \sin\left(\frac{t}{2}\right)\vec{\omega}\right)$$

where the quarternian is expressed in "vector form," consisting of *real part*, q_R, and vector part, $\vec{q}_V : q = (q_R, \vec{q}_V)$. Hence,

$$\dot{q} \cdot q^* = \left(-\frac{1}{2}\sin\left(\frac{t}{2}\right), \frac{1}{2}\vec{\omega}\cos\left(\frac{t}{2}\right)\right)\left(\cos\left(\frac{t}{2}\right), -\sin\left(\frac{t}{2}\right)\vec{\omega}\right)$$

Using the rule for multiplication of quaternions in vector form [1] yields

$$\dot{q} \cdot q^* = (0, \frac{1}{2}\vec{\omega}) .$$

Problem # 2: (15 points, Problem 11(a,b,d) in Chapter 2 of the MLS text)

[1] if quaternians $q = (q_R, \vec{q}_V)$ and $p = (p_R, \vec{p}_V)$ are multiplied, then the resulting quaternian in vector form is: $q \cdot p = (q_R p_R - \vec{q}_V \cdot \vec{p}_V, p_R\vec{q} + q_R\vec{p}_V - \vec{q}_V \times \vec{p}_V)$.

Part (a): Recall that the matrix exponential of a twist, $\hat{\xi}$, is:

$$e^{\phi\hat{\xi}} = I + \frac{\phi}{1!}\hat{\xi} + \frac{\phi^2}{2!}\hat{\xi}^2 + \frac{\phi^3}{3!}\hat{\xi}^3 + \cdots$$

First, let's consider the case of $\xi = (v, \omega)$, with $\omega = 0$. If:

$$\hat{\xi} = \begin{bmatrix} 0 & 0 & v_x \\ 0 & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

then $\hat{\xi}^2 = 0$. Thus

$$e^{\phi\hat{\xi}} = \begin{bmatrix} 1 & 0 & \phi v_x \\ 0 & 1 & \phi v_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & \vec{v}\phi \\ \vec{0}^t & 1 \end{bmatrix}$$

To compute the exponential for the more general case in which $\omega \neq 0$, let us assume that $||\omega|| = 1$. In this case, note that $\hat{\omega}^2 = -I$, where I is the 2×2 identity matrix. It is easiest if we choose a different coordinate system in which to perform the calculations. Let

$$\hat{\xi} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \vec{v} \\ \vec{0}^T & 0 \end{bmatrix}$$

Let

$$g = \begin{bmatrix} I & \hat{\omega}\vec{v} \\ \vec{0}^T & 1 \end{bmatrix}$$

Let is define a new twist, $\hat{\xi}'$:

$$\begin{aligned} \hat{\xi}' &= g^{-1}\hat{\xi}g \\ &= \begin{bmatrix} I & -\hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & \vec{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & \hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\omega} & (\hat{\omega}^2\vec{v} + \vec{v}) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where we made use of the identity $\hat{\omega}^2 = -I$. That is, we have chosen a coordinate system in which $\hat{\xi}'$ corresponds to a pure rotation. Thus,

$$e^{\phi\hat{\xi}'} = \begin{bmatrix} e^{\phi\hat{\omega}} & 0 \\ 0 & 1 \end{bmatrix}.$$

Using Eq. (2.35) on page 42 of the MLS text:

$$e^{\phi\hat{\xi}} = g e^{\phi\hat{\xi}'} g^{-1} = \begin{bmatrix} e^{\phi\hat{\omega}} & (I - e^{\phi\hat{\omega}})\hat{\omega}\vec{v}\phi \\ 0 & 1 \end{bmatrix}$$

which is clearly an element of $SE(2)$.

Part(b): It is easy to see from part (a) that the twist $\xi = (v_x, v_y, 0)^T$ maps directly to the planar translation (v_x, v_y).

The twist corresponding to pure rotation about a point $\vec{q} = (q_x, q_y)$ can be thought of as the Ad-transformation of a twist, $\xi' = (0, 0, \omega)$, which is pure rotation, by a transformation, g, which is pure translation by $\vec{q}$:

$$\xi = \mathrm{Ad}_h \xi' = (h\hat{\xi}' h^{-1})^\vee \tag{1}$$

where

$$h = \begin{bmatrix} I & \vec{q} \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \hat{xi}' = \begin{bmatrix} \hat{\omega} & 0 \\ \vec{0}^T & 0 \end{bmatrix}.$$

Expanding Eq. (1) gives:

$$\xi = (h\hat{\xi}' h^{-1})^\vee = \begin{bmatrix} \hat{\omega} & -\hat{\omega}\vec{q} \\ \vec{0}^T & 0 \end{bmatrix}^\vee = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix}$$

assuming $\omega = 1$.

Problem #3: (10 points). Problem 7 in Chapter 2 of MLS.

We can use the "particle counting" argument used in a previous homework, and also used in class. In this approach, a rigid body in n-dimensional Euclidean space is made up of particles, where each particle has n DOF when it is not constrained to be in a rigid body. The key thing to recognize is the number of constraints needed to join the particles to make a rigid body. For a particle in n dimensional space, the total number of DOF for N particles, which are not constrained to be a rigid body, is nN. The first particle, P_1, has no constraints on its motion. Particle P_2 has one constraint on it's location to be joined to the rigid body, etc. Partial P_n has $(n-1)$ constraints. Particles $P_{n+1}, \ldots, P_N$ have n constraints. So, the total DOF has of the rigid body is the sum of DOF of all particles without constraints, minus the number of constraints:

$$Nn \;-\; [(N-n)n - \sum_{i-1}^{n}(n-i) \;=\; n^2 - \sum_{i=1}^{n}(n-i) = n^2 \;-\frac{1}{2}(n^2+n) = \frac{1}{2}(n^2+n)$$

Problem #3: (15 points)

Part (a): Elements of $SU(2)$ have the form:

$$\begin{bmatrix} \mathbf{z} & \mathbf{w} \\ -\mathbf{w}^* & \mathbf{z}^* \end{bmatrix} = \begin{bmatrix} (a+ib) & (c+id) \\ -(c-id) & (a-ib) \end{bmatrix}$$

where $zz^* + ww^* = a^2 + b^2 + c^2 + d^2 = 1$. To show that the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

form a basis for $SU(2)$, let A, B, C, and D be real numbers. Then, the matrix formed by the product of A, B, C,and D with these matrices is:

$$A\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+B\begin{bmatrix}i & 0\\0 & -i\end{bmatrix}+C\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}+D\begin{bmatrix}0 & i\\i & 0\end{bmatrix}=\begin{bmatrix}A+iB & C+iD\\C-iD & A-iB\end{bmatrix}$$

is a matrix in $SU(2)$ for any choice of A, B, C, and D where $A^2+B^2+C^2+D^2=1$. Thus these four basis matrices for $SU(2)$ are in 1-to-1 correspondence with the 1, i, j, and k basis elements for the quaternions. Thus, the scalar elements A, B, C, and D are in one-to-one correspondence with the scalar elements of unit quaternions. That is, let a unit quaternion be represented by $q=\lambda_1+\lambda_2 i+\lambda_3 j+\lambda_4 k=(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$. The correspondence is then:

$$\lambda_1 = A=Re(z)=\frac{z+z^*}{2} \quad (2)$$

$$\lambda_2 = B=Im(z)=\frac{i(z^*-z)}{2} \quad (3)$$

$$\lambda_3 = C=Re(w)=\frac{w+w^*}{2} \quad (4)$$

$$\lambda_4 = D=Im(w)=\frac{i(w^*-w)}{2} \quad (5)$$

Part (b): The unit quaternion elements are in one-to-one correspondence with the Euler parameters of a rotation: $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=(\cos\frac{\phi}{2},\omega_x\sin\frac{\phi}{2},\omega_y\sin\frac{\phi}{2},\omega_z\sin\frac{\phi}{2})$. ϕ is the rotation about an axis represented by a unit vector $\vec{\omega}=[\omega_x\ \omega_y\ \omega_z]^T$. A 2×2 complex matrix which represents an arbitrary rotation as a function of the z-y-x Euler angles can be developed as the product of 2×2 complex matrices which represent rotations about the z, y, and x axes. A rotation about the x-axis of amount γ has the 2×2 representation (since $\lambda_1=\cos\frac{\gamma}{2}$, $\lambda_2=\sin\frac{\gamma}{2}$, $\lambda_3=\lambda_4=0$):

$$\begin{bmatrix}(\cos\frac{\gamma}{2}+i\sin\frac{\gamma}{2}) & 0\\0 & (\cos\frac{\gamma}{2}-i\sin\frac{\gamma}{2})\end{bmatrix}=\begin{bmatrix}e^{i\frac{\gamma}{2}} & 0\\0 & e^{-i\frac{\gamma}{2}}\end{bmatrix}$$

Similarly, a rotation of amount ϕ about the y-axis can be represented as:

$$\begin{bmatrix}\cos\frac{\phi}{2} & \sin\frac{\phi}{2}\\-\sin\frac{\phi}{2} & \cos\frac{\phi}{2}\end{bmatrix}$$

while a rotation of amount ψ about the z-axis can be represented as:

$$\begin{bmatrix}\cos\frac{\psi}{2} & i\sin\frac{\psi}{2}\\i\sin\frac{\psi}{2} & \cos\frac{\psi}{2}\end{bmatrix}$$

The product of these matrices yields the result.

Part (c):

$$\phi = 2\cos^{-1}(a) = 2\cos^{-1}(\frac{\mathbf{z}+\mathbf{z}^*}{2}) \tag{6}$$

$$\omega_x = \frac{b}{\sqrt{b^2+c^2+d^2}} = \frac{(\mathbf{z}-\mathbf{z}^*)/2}{\sqrt{(\frac{\mathbf{z}-\mathbf{z}^*}{2})^2+\mathbf{ww}^*}} \tag{7}$$

$$\omega_y = \frac{c}{\sqrt{b^2+c^2+d^2}} = \frac{(\mathbf{w}+\mathbf{w}^*)/2}{\sqrt{(\frac{\mathbf{z}-\mathbf{z}^*}{2})^2+\mathbf{ww}^*}} \tag{8}$$

$$\omega_z = \frac{d}{\sqrt{b^2+c^2+d^2}} = \frac{(\mathbf{w}-\mathbf{w}^*)/2}{\sqrt{(\frac{\mathbf{z}-\mathbf{z}^*}{2})^2+\mathbf{ww}^*}} \tag{9}$$

Problem 4: (10 points) Let Z-X-Y Euler angles be denoted by ψ, ϕ, and γ. First let's develop an expression for the net rotation due to Z-Y-X rotations. First find expressions for each of the individual rotation matrices:

$$R_\psi = Rot(\vec{z},\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_\phi = Rot(\vec{y},\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$R_\gamma = Rot(\vec{x},\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

where the notation $R(\vec{p},\alpha)$ means rotation by angle α about axis $\vec{p}$. The final expression is obtained by multiplying these matrices:

$$R(\psi,\phi,\gamma) = R_\psi R_\phi R_\gamma = \begin{bmatrix} c\psi c\phi & c\psi s\phi s\gamma - s\psi c\gamma & c\psi s\phi c\gamma + s\psi s\gamma \\ s\psi c\phi & s\psi s\phi s\gamma + c\psi c\gamma & s\psi s\phi c\gamma - c\psi s\gamma \\ -s\phi & c\phi s\gamma & c\phi c\gamma \end{bmatrix}$$

where $s\psi = \sin\psi$, $c\phi = \cos\phi$, etc.

To find an expression for the angles ψ, ϕ, and γ as a function of the a_{ij} in the matrix

$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

note that $\sin\phi = -a_{31}$. There are two solutions to this equation: $\phi_1 = \sin^{-1}(a_{31})$ and $\phi_2 = \pi - \phi_1$. Manipulation of the (a_{11}, a_{21}) and (a_{32}, a_{33}) terms yields:

$$\psi = Atan2[\frac{a_{21}}{\cos\phi}, \frac{a_{11}}{\cos\phi}]; \qquad \gamma = Atan2[\frac{a_{32}}{\cos\phi}, \frac{a_{33}}{\cos\phi}].$$