ME 115(a): Solution to Homework #5

Problem 1: (10 points) Find the Denavit-Hartenberg parameters for manipulators (i), (ii), and (iv) in Figure 3.23 of the MLS text

(i) The choice of the stationary frame is abitrary. For simplicity, place the origin of the stationary frame at the point where all three revolute joints intersect. Place the z-axis of the stationary frame, z_S, collinear with the first joint axis. Orient the x-axis of the stationary frame to be orthogonal to both joint axes 1 and 2 (pointing toward the right in Figure 3.23(i)). Similary, there are many choices for the tool frame. Let's assume that the tool frame is coincident with the link frame of link 3, as determined using the Denavit-Hartenberg procedure. Then, the D-H parameters are:

$$\begin{array}{cccc} a_0 = 0 & \alpha_0 = 0 & d_1 = 0 & \theta_1 = \text{ variable} \\ a_1 = 0 & \alpha_1 = -\frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{ variable} \\ a_2 = 0 & \alpha_2 = \frac{\pi}{2} & d_3 = 0 & \theta_3 = \text{ variable} \\ a_3 = 0 & \alpha_3 = 0 & d_4 = 0 & \theta_4 = \text{ constant} = 0 \end{array}$$

(ii) The choice of the stationary frame is abitrary. Place its origin along joint axis 1, but not necessarily at the point of coincidence of joint axes 1 and 2. The tool frame origin is placed in the middle of the "U", with its x-axis collinear with the mechanical link axis, and with its z-axis parallel to joint axis 2. In this case,

$$\begin{array}{cccc} a_0 = 0 & \alpha_0 = 0 & d_1 \neq 0 & \theta_1 = \text{ variable} \\ a_1 = 0 & \alpha_1 = \frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{ variable} \\ a_2 \neq 0 & \alpha_2 = -\frac{\pi}{2} & d_3 \neq 0 & \theta_3 = \text{ variable} \\ a_3 \neq 0 & \alpha_3 = \frac{\pi}{2} & d_4 = 0 & \theta_3 = \text{ constant} \end{array}$$

where the value of d_1 will be determined by the location of the stationary frame origin.

(iv) The choice of the stationary frame is abitrary. For simplicity, place the origin of the stationary frame at the point where all three joints intersect. Place the z-axis of the stationary frame, z_S, collinear with the first joint axis. Orient the x-axis of the stationary frame to be orthogonal to both joint axes 1 and 2 (pointing toward the right in Figure 3.23(i)). Similary, there are many choices for the tool frame. Let's assume that the tool frame is parallel with the link frame of link 3, (as determined using the Denavit-Hartenberg procedure), but its origin lies at the tip of the mechanism (in the "U" of Figure 3.23(iv)). Then, the D-H parameters are:

$$\begin{array}{cccc} a_0 = 0 & \alpha_0 = 0 & d_1 = 0 & \theta_1 = \text{ variable} \\ a_1 = 0 & \alpha_1 = -\frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{ variable} \\ a_2 = 0 & \alpha_2 = \frac{\pi}{2} & d_3 = \text{variable0} & \theta_3 = \ 0 \\ a_3 = 0 & \alpha_3 = 0 & d_4 = \text{constant} & \theta_3 = \text{ constant} = 0 \end{array}$$

where the constant d_4 depends upon the offset between the origin of link frame 3 and the origin of the tool frame.

Problem 2: (30 points) Consider the simple manipulator (iii) associated with Prob.4 in Chapter 3 of the MLS text.

- **Part (a):** *Determine the forward kinematics using the Denavit-Hartenberg approach.*

  First, let's find the D-H parameters. For simplicity, let us choose the z-axis of the stationary frame to be collinear with the first joint axis. The origin of the stationary frame is located some distance below the point of intersection of the first two axes. Also, choose the tool frame origin to coincide with the intersection point of the last three axes (the "wrist"). Also, assume that the link 6 frame of the Denavit-Hartenberg approach is the tool frame. While you were only asked to find the forward kinematics for the first 3 joints, here are the D-H parameters for all links/joints:

$$\begin{array}{llll} a_0 = 0 & \alpha_0 = 0 & d_1 \neq 0 & \theta_1 = \text{ variable} \\ a_1 = 0 & \alpha_1 = \frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{ variable} \\ a_2 = 0 & \alpha_2 = -\frac{\pi}{2} & d_3 = \text{variable}0 & \theta_3 = 0 \text{ (constant)} \\ a_3 = 0 & \alpha_3 = \frac{\pi}{2} & d_4 = 0 & \theta_4 = \text{ variable} \\ a_4 = 0 & \alpha_3 = \frac{\pi}{2} & d_5 = 0 & \theta_5 = \text{ variable} \\ a_5 = 0 & \alpha_3 = \frac{\pi}{2} & d_6 = 0 & \theta_6 = \text{ variable} \\ a_6 = 0 & \alpha_3 = 0 & d_7 = 0 & \theta_7 = 0 \end{array}$$

  To find the forward kinematics using the Denavit-Hartenberg approach, one can use the formula

$$g_{ST}(\vec{\theta}) = g_{S1}(\theta_1)g_{12}(\theta_2)g_{23}(d_3)g_{34}(\theta_4)g_{45}(\theta_5)g_{56}(\theta_6)g_{6T}.$$

  where each $g_{i,i+1}$ is given by:

$$\begin{bmatrix} \cos\theta_{i+1} & -\sin\theta_{i+1}0 & a_i & \\ \sin\theta_{i+1}\cos\alpha_i & \cos\theta_{i+1}\cos\alpha_i & -\sin\alpha_i & -d_{i+1}\sin\alpha_i \\ \sin\theta_{i+1}\sin\alpha_i & \cos\theta_{i+1}\sin\alpha_i & -\cos\alpha_i & d_{i+1}\cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  Plugging in the D-H parameters from above yields:

$$g_{ST} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & \\ 0 & 0 & 1 & 0 \\ -\sin\theta_2 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & 01 & \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} \cos\theta_4 & -\sin\theta_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_4 & \cos\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_5 & -\sin\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_5 & \cos\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} g_{6T}$$

  The forward kinematics of the first 3 joints (i.e., to the wrist center) is:

$$g_{ST}(\theta_1, \theta_2, d_3) = \begin{bmatrix} c_1c_2 & -s_1 & -c_1s_2 & -d_3c_1s_2 \\ s_1c_2 & -c_1 & -s_1s_2 & -d_3s_1s_2 \\ -s_2 & 0 & -c_2 & d_1 - d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  where $s_k = \sin(\theta_k)$ and $c_j = \cos(\theta_j)$.

- **Part(b):** In the Product of Exponentials (POE) approach, the forward kinematics is given by:
$$g_{ST}(\theta_1, \theta_2, d_3) = e^{\theta_1 \hat{\xi}_1} e^{\theta_2 \hat{\xi}_2} e^{d_3 \hat{\xi}_3} g_{ST}(0)$$
where $\vec{\xi}_1$, $\vec{\xi}_2$, $\vec{\xi}_3$ are the twists associated with the joint axes 1, 2, and 3, as described in the "zero" or "home" position of the manipulator. Take the configuration in the text as the zero reference configuration, and place the stationary frame at the same location. Then the twists of the first six joint axes are:
$$\vec{\xi}_1 = \begin{bmatrix} v_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} h_1\omega_1 + \rho_1 \times \omega_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{z}_S \end{bmatrix}$$
$$\vec{\xi}_2 = \begin{bmatrix} v_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} h_2\omega_2 + \rho_2 \times \omega_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} -d_1\vec{z}_S \times \vec{y}_S \\ -\vec{y}_S \end{bmatrix} = \begin{bmatrix} d_1\vec{x}_S \\ -\vec{y}_S \end{bmatrix}$$
$$\vec{\xi}_3 = \begin{bmatrix} \omega_3 \\ \vec{0} \end{bmatrix} = \begin{bmatrix} \vec{x}_S \\ \vec{0} \end{bmatrix}$$
$$\vec{\xi}_4 = \begin{bmatrix} v_4 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} h_4\omega_4 + \rho_4 \times \omega_4 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} -(d_1\vec{z}_S + d_3\vec{x}_S) \times \vec{y}_S \\ -\vec{y}_S \end{bmatrix} = \begin{bmatrix} d_1\vec{x}_S - d_3\vec{z}_S \\ -\vec{y}_S \end{bmatrix}$$
$$\vec{\xi}_5 = \begin{bmatrix} v_5 \\ \omega_5 \end{bmatrix} = \begin{bmatrix} h_5\omega_5 + \rho_5 \times \omega_5 \\ \omega_5 \end{bmatrix} = \begin{bmatrix} d_3\vec{x}_S \times \vec{z}_S \\ \vec{z}_S \end{bmatrix} = \begin{bmatrix} -d_3\vec{y}_S \\ \vec{z}_S \end{bmatrix}$$
$$\vec{\xi}_6 = \begin{bmatrix} v_6 \\ \omega_6 \end{bmatrix} = \begin{bmatrix} h_6\omega_6 + \rho_6 \times \omega_6 \\ \omega_6 \end{bmatrix} = \begin{bmatrix} d_1\vec{z}_S \times \vec{x}_S \\ \vec{x}_S \end{bmatrix} = \begin{bmatrix} -d_1\vec{y}_S \\ \vec{x}_S \end{bmatrix}$$
where $\vec{x}_S$, $\vec{y}_S$, and $\vec{z}_S$ are the basis vectors of the stationary frame, and assume values
$$\vec{x}_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{y}_S = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{y}_S = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$
The forward kinematics for the first 3 joints (to the wrist point) is then:
$$g_{ST} = e^{\theta_1 \hat{\xi}_1} e^{\theta_2 \hat{\xi}_2} e^{d_3 \hat{\xi}_3} g_{ST}(0)$$
with $g_{ST}(0)$ taking the form:
$$\begin{bmatrix} 1 & 0 & 0 & d_3 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- **Part (c):** You can take either a geometric or algebraic approach to this problem. We'll take an algebraic approach, since the algebra is so simple. The goal is to find the solution to the following three equation:
$$\begin{bmatrix} x_D \\ y_D \\ z_D \end{bmatrix} = \begin{bmatrix} -d_3c_1s_2 \\ -d_3s_1s_2 \\ d_1 - d_3c_2 \end{bmatrix}.$$

Squaring the first two equations and adding yields:

$$d_3^2 s_2^2 = x_D^2 + y_D^2 \tag{1}$$

From the third equation we get:

$$(z_D - d_1)^2 = d_3^2 c_2^2 \ . \tag{2}$$

Adding Equations (1) and (2) yields:

$$d_3^2 = x_D^2 + y_D^2 + (z_D - d_1)^2,$$

from which we obtain two solutions:

$$d_3 = \pm\sqrt{x_D^2 + y_D^2 + (z_D - d_1)^2} \ . \tag{3}$$

For each unique value of d_3 we can solve Equation (1) and the z-component equation of the forward kinematics to yield:

$$\theta_2 \ = \ Atan2[\frac{(x_D^2 + y_D^2)^{-1/2}}{d_3}, \frac{d_1 - z_D}{d_3}] \ .$$

Finally, θ_1 can be determined from the first two forward kinematic equations (the x and y components):

$$\theta_1 \ = \ Atan2[\frac{y_D}{-d_3 s_2}, \frac{x_D}{-d_3 s_2}] \ .$$