ME 115(b): Solution to Problem Set #5

Problem #1 (20 points): Problem 10 (b) in Chapter 5 of MLS.

Let's assume that the disk-like finger tips make a frictional point contact with the rectangular grasped object. In general, the grasp constraint for a hand takes the form:

$$J_h(\vec{\theta}, g_{po})\dot{\vec{\theta}} \;=\; G^T(\vec{\theta}, g_{po})V^b_{po}$$

where G is the grasp map, $\vec{\theta}$ is the vector of all finger joint variables, g_{po} is the position of the grasped object relative to the hand, and J_h is the hand Jacobian, which for this two fingered hand takes the form:

$$J_h \;=\; \begin{bmatrix} B^T_{c_1} Ad^{-1}_{g_{s_1,c_1}} J^s_{s_1,f_1}(\vec{\theta}_1) & 0 \\ 0 & B^T_{c_2} Ad^{-1}_{g_{s_2,c_2}} J^s_{s_2,f_2}(\vec{\theta}_2) \quad 0 \end{bmatrix}.$$

Recall that B_{c_i} is the wrench basis of the i^{th} contact. To develop the concrete form of this grasp constraint for the given hand, put a reference frame on the grasped rectangular object at its geometric center. With this choice, the contact normals of the two disk fingers pass through the center of the object reference frame. The wrench bases of each contact are

$$B_{c_i} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad i = 1, 2.$$

For the rough sketch in Problem 10(b) of MLS, let's assume that the finger tip contacts lie directly above the revolute joints which connect each finger to the palm. In thise case

$$g_{s_1,c_1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & h \\ 0 & 0 & 1 \end{bmatrix} \qquad g_{s_2,c_2} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & h \\ 0 & 0 & 1 \end{bmatrix}.$$

where h is the distance between the base revolute joint and the finger contact. Hence, If we let $\theta_{1,1}$ and $\theta_{1,2}$ denote the joint angles of the first finger, and $\theta_{2,1}$ and $\theta_{2,2}$ denote the angles of the second finger, and if the link lengths of the i^{th} finger are respectively $l_{i,1}$ and $l_{i,2}$, then

$$J_h(\vec{\theta}, g_{po})\dot{\vec{\theta}} = \begin{bmatrix} -l_{1,2}\dot{\theta}_{1,2}\sin(\theta_{1,1}+\theta_{1,2}) \\ -h\dot{\theta}_{1,1} - l_{1,2}\cos(\theta_{1,1}+\theta_{1,2}) \\ -l_{2,2}\dot{\theta}_{2,2}\sin(\theta_{2,1}+\theta_{2,2}) \\ -h\dot{\theta}_{2,1} + l_{2,2}\cos(\theta_{2,1}+\theta_{2,2}) \end{bmatrix}$$

If the body velocities of the grasped rectangular object are denoted by $(\dot{x}, \dot{y}, \dot{\phi})$, then:

$$G^T\vec{V}_{po} \;=\; \begin{bmatrix} -\dot{y} + \dot{\phi}w \\ -\dot{x} \\ \dot{y} + \dot{\theta}w \\ \dot{x} \end{bmatrix}$$

where w is the half-width of the rectangle.

Problem #2 (30 points)

Part (a): To show that the tangent at curve parameter t of the evolute of any arbitrary curve $\alpha(t)$ is the normal to α at t, recall that the evolute, $\beta(t)$, of a *planar* curve, $\alpha(t)$, is:

$$\beta(t) = \alpha(t) + \frac{1}{\kappa(t)}\vec{n}(t) \tag{1}$$

where $\kappa(t)$ is the curvature of $\alpha(t)$ and $\vec{n}(t)$ is the unit normal vector of $\alpha(t)$ at t. Let's assume that the curve is arc-length parametrized, i.e., t is the arc-length parameter. The tangent to the evolute is simply derived by taking the derivative of Equation (1).

$$\frac{d\beta}{dt} = \vec{\alpha}^{'} + \frac{d}{dt}\left(\frac{1}{\kappa(t)}\vec{n}(t)\right) = \vec{u} + \frac{\kappa(t)\vec{n}^{'}(t) - \vec{n}(t)\kappa^{'}(t)}{\kappa^2(t)} \tag{2}$$

Recall that for a regular curve, $\vec{\alpha}^{''}(t) = \kappa(t)\vec{n}(t)$.

What is $\vec{n}^{'}(t)$ for a planar curve? Since $\vec{n}(t)$ is a unit length curve, then $\vec{n}^{'}(t)$ must be a vector orthogonal to $\vec{n}(t)$–i.e., a vector in the direction of $\vec{\alpha}^{'}(t)$. Thus, assume that $\vec{n}^{'} = \eta\vec{\alpha}^{'}$, where η is some proportionality constant, which is to be determined.

From the relationship $\vec{\alpha}^{'} \cdot \vec{n} = 0$, we can obtain (by taking derivatives):

$$0 = \vec{\alpha}^{''} \cdot \vec{n} + \vec{\alpha}^{'} \cdot \vec{n}^{'} = \kappa\vec{n} \cdot \vec{n} + \vec{\alpha}^{'} \cdot \vec{n}^{'} = \kappa + \vec{\alpha}^{'} \cdot (\eta\vec{\alpha}^{'}) = \kappa + \eta \tag{3}$$

This result implies that $\eta(t) = -\kappa(t)$, and hence:

$$\vec{n}^{'} = -\kappa\vec{\alpha}^{'}. \tag{4}$$

Substituting these results back into Eq. (2) results in:

$$\frac{d\beta(t)}{dt} = \vec{\alpha}^{'} + \frac{\kappa(t)\vec{n}^{'} - \vec{n}(t)\kappa^{'}}{\kappa^2} = \vec{\alpha}^{'} + \frac{-\kappa^2\vec{\alpha}^{'} - \kappa^{'}\vec{n}}{\kappa^2} = \frac{-\kappa^{'}}{\kappa}\vec{n} \tag{5}$$

Thus, $d\beta/dt$ is a vector in the direction of the normal to $\alpha(t)$.

Part (b): The evolute of a circle is a point.

Part (c,d): To show that the evolute of the involute gear tooth profile

$$x(t) = R(\cos(t) + t\sin(t)) \tag{6}$$
$$x(t) = R(\sin(t) - t\cos(t)) \tag{7}$$

is a circle, we need to compute the Evolute curve (Equation (1)) for the (not necessarily arc-length parameterized) curve (6). Note that

$$\vec{\alpha}^{'} = \begin{bmatrix} R(-\sin(t) + t\cos(t) + \sin(t)) \\ R(\cos(t) + t\sin(t) - \cos(t)) \end{bmatrix} = Rt\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

Hence, $||\vec{\alpha}'|| = Rt$. Also note that for planar curves, the unit normal vector can be found by rotating the unit tangent vector by $\pi/2$. In this case:

$$\vec{n}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

Using the course notes on the planar contact equations, the curvature of a non-arclength-parameterized curve can be found as:

$$\kappa(t) = -\frac{1}{||\vec{\alpha}'||^2}\left(\vec{\alpha}'(t) \cdot \vec{n}'(t)\right)$$

where $'$ denoted differentiation with respect to the curve parameter, which may not be arc length. Substituting in the appropriate terms:

$$\kappa(t) = -\frac{1}{(Rt)^2}\begin{bmatrix} Rt\cos(t) \\ Rt\sin(t) \end{bmatrix} \cdot \begin{bmatrix} -\cos(t) \\ -\sin(t) \end{bmatrix} = \frac{1}{Rt}\ . \tag{8}$$

Thus, the evolute is:

$$\beta(t) = \alpha(t) + \frac{1}{\kappa}\vec{n}(t) = \begin{bmatrix} R(\cos(t) \ + \ t\sin(t)) \\ R(\sin(t) \ - \ t\cos(t)) \end{bmatrix} + Rt\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} = R\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

which is the parametrization of a planar circle with radius R.

Extra Credit: (5 points) A planar ellipse can be easily parametrized as

$$\alpha(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a\cos(t) \\ b\sin(t) \end{bmatrix} \tag{9}$$

where a and b are the major and minor principal dimensions of the ellipse. Show that the evolute of the planar ellipse is an **astroid**,

$$\beta(t) \ = \ \begin{bmatrix} \frac{(a^2-b^2)\cos^3(t)}{a} & \frac{(b^2-a^2)\sin^3(t)}{b} \end{bmatrix} \tag{10}$$

and plot the evolute and the ellipse.