# ME 132

## A Symbolic Formula for a C-obstacle

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# 1 Background

Let the robot $\mathcal{A}$ and the obstacle $\mathcal{O}$ be *planar convex polygons.* Our goal in this handout is to derive a closed-form formula for the boundary of the c-obstacle $\mathcal{CO}$. That is, to find a continuous real-valued function $d(q)$ on c-space, $d : \mathcal{C} \rightarrow I\!R$, such that

$$\mathsf{bdy}\big(\mathcal{CO}\big) = \{q \in \mathcal{C} : d(q) = 0\} \quad \text{where } q = (x, y, \theta).$$

where $\mathsf{bdy}(\mathcal{S})$ denotes the boundary of $\mathcal{S}$. As discussed below, the boundary of $\mathcal{CO}$ consists of smooth two-dimensional patches that meet along curved or straight edges. Let $n_{\mathcal{A}}$ and $n_{\mathcal{O}}$ be the number of vertices of $\mathcal{A}$ and $\mathcal{O}$, respectively. From the formula presented below we shall see that $\mathsf{bdy}(\mathcal{CO})$ consists of $2n_{\mathcal{A}}n_{\mathcal{O}}$ patches. The formula will be union of $2n_{\mathcal{A}}n_{\mathcal{O}}$ terms, each representing one patch. Each patch term will be in turn be defined by the intersection of several inequalities.

Recall that $\mathcal{A}(q)$ is the set of points occupied by the robot when placed at configuration $q$. Each two-dimensional patch comprising the boundary of $\mathcal{CO}$ corresponds to one of two generic types of contact.

- **Type EV contact:** Configurations $q$ where an edge of the robot, denoted $E_i^{\mathcal{A}}(q)$, contacts a fixed vertex $o_j$ of the obstacle $\mathcal{O}$. These contacts are often called *Type A* contacts in the robotics motion planning literature, in deference to nomenclature use in the original paper on configuration space by Lozano-Perez, Mason, and Taylor.

- **Type VE contact:** Configurations $q$ where a vertex $a_i(q)$ of $\mathcal{A}$ contacts a fixed edge $E_j^{\mathcal{O}}$ of $\mathcal{O}$. These contacts are often called *Type B* contacts in the literature.

The fixed-$\theta$ slice of each patch is a straight-line segment when both robot and obstacle are polygons. As the robot rotates, the segments of a Type EV patch rotate (and therefore a Type EV patch is a "ruled surface"), and the segments of a Type VE patch translate parallel to the associated edge of $\mathcal{O}$.

The curves separating neighboring patches are of two types. Those that lie in a fixed-$\theta$ slice correspond to edge-edge contact. Those that lie transversally to the fixed-$\theta$ slices correspond to vertex-vertex contact. The vertex-vertex ones have the property that their projection onto the $(x, y)$-plane is a circular arc. See Figure 1.

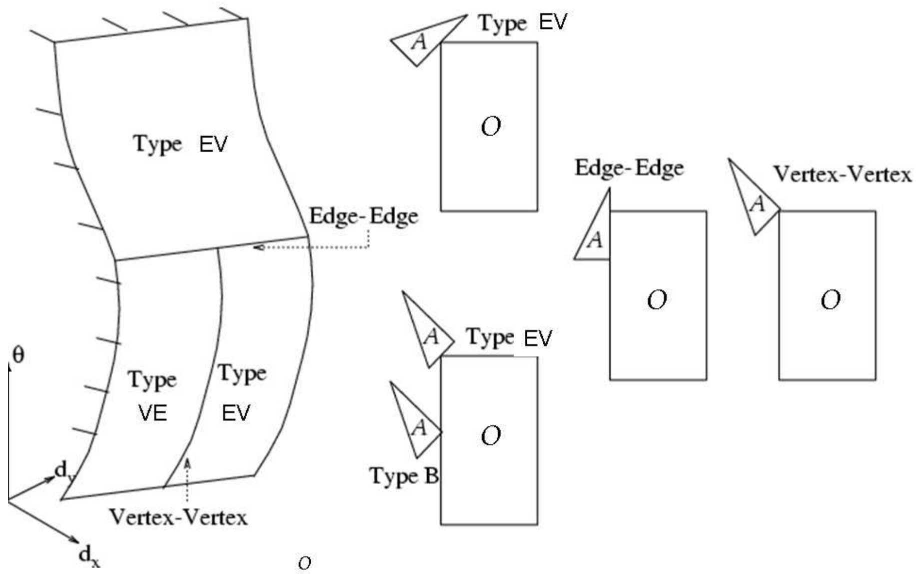


Figure 1: Type EV and Type VE patches

# 2 The Formula

First we derive a formula for a Type EV patch. Consider the set of all configurations $q$ where the edge $E_i^{\mathcal{A}}(q)$ touches the vertex $o_j$, such that the interiors of $\mathcal{A}$ and $\mathcal{O}$ do not overlap (Figure 2(a)). Let $l$ be the line underlying the edge $E_i^{\mathcal{A}}(q)$. The equation of $l$ is $\nu_i^{\mathcal{A}}(q) \cdot (x - a_i(q)) = 0$. The first constraint is that $o_j$ must lie on $l$:

$$\nu_i^{\mathcal{A}}(q) \cdot (o_j - a_i(q)) = 0. \tag{1}$$

This defines the smooth two-dimensional surface underlying the patch. Second, regardless of where $o_j$ lies on $l$, the range of allowed orientations for $\mathcal{A}$ such that its interior does not overlap the interior of $\mathcal{O}$ is determined as follows. The allowed orientations are exactly those were the vertices $o_{j-1}$ and $o_{j+1}$ lie in the halfspace bounded by $l$ which does not contain $\mathcal{A}$. In formulas:

$$\nu_i^{\mathcal{A}}(q) \cdot (o_{j-1} - a_i(q)) \geq 0$$

and

$$\nu_i^{\mathcal{A}}(q) \cdot (o_{j+1} - a_i(q)) \geq 0.$$

The intersection of these inequalities with (1) gives the "upper" and "lower" edges of the patch, corresponding to edge-edge contacts. Last, $o_j$ must lie within the edge $E_i^{\mathcal{A}}(q)$. Convince yourself that this is captured by the following two inequalities:

$$(o_j - a_i(q)) \cdot (a_{i+1}(q) - a_i(q)) \geq 0$$

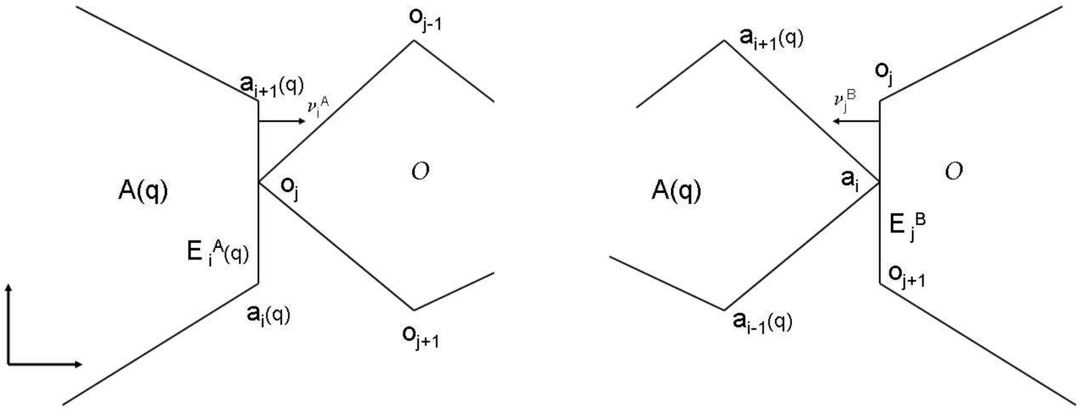


Figure 2: Type EV contact (left), and Type VE contact (right)

and

$$(o_j - a_i(q)) \cdot (a_{i+1}(q) - a_i(q)) \leq \|E_i^{\mathcal{A}}\|^2.$$

($\|E_i^{\mathcal{A}}\|$ is the length of the edge $E_i^{\mathcal{A}}(q)$. It does not depend on $q$). These inequalities determine the "left" and "right" edges of the patch, corresponding to vertex-vertex contacts. The patch is described by the conjunction (equivalently, intersection of the respective sets) of these five terms.

Consider now a Type VE patch, where the vertex $a_i(q)$ touches the edge $E_j^O$, such that the interiors of $\mathcal{A}$ and $\mathcal{O}$ do not overlap (Figure 2(b)). Convince yourself that the conjunction of the following five terms exactly characterizes the patch:

$$\nu_j^{\mathcal{O}} \cdot (a_i(q) - o_j) = 0,$$

which is the surface underlying the patch,

$$\nu_j^{\mathcal{O}} \cdot (a_{i-1}(q) - o_j) \geq 0 \quad \text{and} \quad \nu_j^{\mathcal{O}} \cdot (a_{i+1}(q) - o_j) \geq 0,$$

which are the upper and lower edges,

$$(a_i(q) - o_j) \cdot (o_{j+1} - o_j) \geq 0, \quad \text{and} \quad (a_i(q) - o_j) \cdot (o_{j+1} - o_j) \leq \|E_j^{\mathcal{O}}\|^2,$$

which are the left and right edges.

In total we have

$$\mathsf{bdy}(\mathcal{CO}) = \left( \bigcup_{1 \leq i \leq n_{\mathcal{A}}, 1 \leq j \leq n_{\mathcal{O}}} \big(\text{Type EV patch}\big) \right) \bigcup \left( \bigcup_{1 \leq i \leq n_{\mathcal{A}}, 1 \leq j \leq n_{\mathcal{O}}} \big(\text{Type VE patch}\big) \right). \quad (2)$$

where the shorthand notation “Type EV patch” stands for the set of equalities and inequalities that define a Type EV patch. There are exactly $2n_{\mathcal{A}}n_{\mathcal{O}}$ terms, which is the number of distinct patches comprising the boundary of $\mathcal{CO}$.

The continuous function, $d(q)$, that implicitly describes the boundary can be defined as follows. Given two sets described by $\mathcal{S}_1 = \{q : \alpha(q) \leq 0\}$ and $\mathcal{S}_2(q) = \{q : \beta(q) \leq 0\}$, $\mathcal{S}_1 \bigcap \mathcal{S}_2$ can be replaced by the inequality $\max\{\alpha(q), \beta(q)\} \leq 0$ and $\mathcal{S}_1 \bigcup \mathcal{S}_2$ can be replaced by $\min\{\alpha(q), \beta(q)\} \leq 0$. Since a patch is defined as the intersection of inequalities and the c-osbstacle boundary is defined as the union of patches, Equation (2) can be constructed from nested sequences of min and max operations.