CDS 110(b) Final Exam

(Winter 2012/2013)

Instructions

1. Limit your total time to 5 hours. It is okay to take a break in the middle of the exam if you need to ask the Instructor or TA a question, or to go to dinner. If you run out of time, indicate how you would proceed as explicitly as possible.
2. You may use any class notes, books, or other written material posted on the course web site. You may not discuss this final with other class students or other people except me or the class Teaching Assistants.
3. You may use Mathematica, MATLAB, or any software or computational tools to assist you.
4. You cannot use the internet to solve these problems, except for material on the course web site.
5. The final is nominally due by 5:00 p.m. on the last day of finals. However, since the final was posted slightly behind the original schedule, turning it in up to noon on the next day after the end of finals is fine if students need that extra time.
6. The point values are listed for each problem to assist you in allocation of your time.

Problem 1: The goal of preserving fuel use is critical for spacecraft design and deployment. In this problem you will consider a highly simplified version of a satellite attitude control problem. Assume a single rigid body satellite free-floating above earth is constrained to move in a plane. In this simplified model, θ is the angle which describes the satellite's orientation in that plane. The dynamics which relate the control input, u to the satellite's orientation are:

$$I\ddot{\theta} = u \tag{1}$$

where I is the rotational inertia of the satellite, and u can be interpreted as the torque applied to the satellite. In this problem, the torque is provided by a *thruster*. For example, a compressed gas canister can provide thrust, and the loss of gas is roughly proportional to the amount of thrust generated. Note that when the gas is all used up, reorientation is no longer possible, and so it is essential to minimize gas usage by minimizing the thrust used to carry out a maneuver.

Assume that the spacecraft must carry out a reorientation maneuver, starting from an initial orientation θ_0 at time $t_0 = 0$, and ending at a final orientation, θ_f, at $t_f = T$. Also assume that the control is bounded: $|u| \leq 1$.

Part (a): (15 points) When a gas thruster is used , one can approximately model the total cost as:

$$J(u) = \int_0^T (1 + \alpha\,|u|)dt \tag{2}$$

where α is a positive constant. Note that the constant "1" in the cost is a minimal time function, while the term $\alpha|u|$ is proportional to fuel use. Thus, this cost function trades off between minimum time and fuel use, with α begin the parameter that trades-off these two goals.

Show that the optimal control solution for this cost is a bang-deadzone-bang solution, and write the switching conditions.

Part (b): (5 points) Sketch qualitatively the solution of part (a) for a nonzero value of $\alpha = 1$. Here, the term "solution" means the control as a function of time, the satellite orientation as a function of time, and the satellite rotation rate as a function of time.

Part (c): LQG (15 points) As an alternative to a gas canister, one could use an *ion thruster*, whose energy can be continually replenished by a solar panel. In this case, the use of the thruster is still costly, but it must be balanced against performance. Design a **steady-state** LQG controller, assuming that the cost is:

$$J = E\left\{ \int_0^\infty [x^T Q x + u^T R u] dt \right\}$$

where the state[1] x consists of:

$$x = \begin{bmatrix} (\theta - \theta_f) \\ \dot{\theta} \end{bmatrix}$$

where the weighting matrices Q and R take the form:

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \qquad R = 1$$

[1]Note that the state can be written simply as $x = [\theta\ \dot{\theta}]^T$ if the θ-coordinates are adjusted so that $\theta_f = 0$.

and where a small disturbance due to a *gravity gradient* acts on the system in the following way

$$I\ddot{\theta} \;=\; u + \eta(t) \tag{3}$$

where $\eta(t)$ can be modeled by zero mean white Gaussian noise with covariance 0.001. Assume that only satellite orientation measurements are available for the estimator:

$$y(t) = \theta(t) + \omega(t)$$

and assume the measurement noise $\omega(t)$ is zero mean white Gaussiam with 0.5 degrees variance.

Note: this problem is given in a continous time fashion. If you'd prefer to design both controller and estimator (or just estimator) in discrete time, that is fine.

Part(d): (5 points) Compute the closed loop poles of the controller, and also the closed loop poles of the estimator.

Problem 2: (Fixed Point Smoother) (15 points)
During the course we studied in detail the *fixed lag smoother*. We briefly defined the *fixed point smoother*, but did not study it in detail. In this problem you will analyze a fixed point smoother for the initial state of the dynamical system (which is the discretization of a 1-dimensional second order mass-spring system):

$$\begin{bmatrix} z_{k+1} \\ \dot{z}_{k+1} \end{bmatrix} \;=\; \begin{bmatrix} 1 & \delta t \\ -\beta\ \delta t/m & 1 \end{bmatrix} \begin{bmatrix} z_k \\ \dot{z}_k \end{bmatrix} + \begin{bmatrix} 0 \\ \delta t/m \end{bmatrix} u \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \eta_k \tag{4}$$

where η_k is white, zero mean, Gaussian noise with covariance $Q_k = 0.1$, and $\delta t = 0.05$ is the sampling time of the model discretization. Assume that $m = \beta = 1$. Also assume that the position, z, is measured with a noisy sensor, so that

$$y_k \;=\; z_k \;+\; \omega_k \;=\; \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_k \\ \dot{z}_k \end{bmatrix} + \omega_k \tag{5}$$

where ω_k is a zero mean, white, Gaussian noise with covariance R_k=0.05.

Assume that at time t_0 the initial state of the system is an uncertain variable with mean and covariance:

$$E\begin{bmatrix} z_0 \\ \dot{z}_0 \end{bmatrix} = \begin{bmatrix} \bar{z}_0 \\ \bar{v}_0 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.25 \end{bmatrix} \qquad P_0 = \begin{bmatrix} 1.0 & 0 \\ 0 & 0.5 \end{bmatrix}\ . \tag{6}$$

Recall that a fixed point smoother computes the estimate $\hat{x}_{j|k}$ of a state x at times $k = j, j+1, j+2, \ldots$. Assuming that $u = 0$, compute the **variance** of the initial position estimate $\hat{z}_{0|k}$ for $k = 0, 1, 2, 3, 4, 5$.

Problem 3: (Correlated Noise in Kalman Filtering) (10 points)

Consider the standard linear dynamical system set-up:

$$\begin{aligned} x_{k+1} &= Fx_k + G\eta_k \\ y_{k+1} &= Hx_{k+1} + \omega_k \end{aligned}$$

with Gaussian, zero mean white process and measurement noises:

$$E\begin{bmatrix}\eta_k\\ \omega_k\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

but now there is correlation between the processes and measurement noise:

$$E\left\{\begin{bmatrix}\eta_k\\ \omega_k\end{bmatrix}\begin{bmatrix}\eta_l^T & \omega_l^T\end{bmatrix}\right\} = \begin{bmatrix}Q & S\\ S^T & R\end{bmatrix}\delta_{kl}$$

where δ_{kl} is the Kronecker delta function.

Derive an expression for the measurement update covariance in the case of correlatd process and measurement noise.