ME 72: Engineering Design Laboratory
Analysis of a Differential Drive Vehicle

1 Introduction

These notes are a companion to the in-class lectures on the *differential drive* ground vehicle. Figure 1 shows both a top view and a side view of an idealized differential drive vehicle.

Figure 1: Schematic diagram of an idealized differential drive vehicle. **Left:** top view of the vehicle. **Right:** Side view of the vehicle.

The goal of these notes is to derive *parametric models* that:

- relate the wheel speed to vehicle speed (as a function of key parameters).
- relate wheel torques to vehicle accelerations (as a function of key parameters).

1.1 Basic Kinematic Principles

To derive our results, we will use some very basic principles of rigid body kinematics. Consider a moving rigid body. Attach a reference frame,$\mathcal{F}_2$ (with unit basis vectors $\{x_2, y_2, z_2\}$), to the moving body. Assume that there is an additional, fixed observing frame $\mathcal{F}_1$ (whose unit basis vectors $\{x_1, y_1, z_1\}$). We will use the basic notions of coordinate transformations between points in the moving and fixed reference frames, as well as the notion of the velocity of points in the moving body.

Coordinate Transformations: The transformation between the coordinates of a point P in the moving rigid body (as described by an observer in $\mathcal{F}_2$) to its equivalent representation in the fixed observing reference frame $\mathcal{F}_1$ is:

$$\vec{z} = \vec{d} + R(\theta)\vec{r} \tag{1}$$

where $\vec{d}$ is the vector from the origin of $\mathcal{F}_1$ to the origin of the $\mathcal{F}_2$, and where $\vec{r}$ is the vector to point P, as described in a reference frame $\mathcal{F}_2$. The vector $\vec{z}$ points from the origin of $\mathcal{F}_1$ frame to that same point P, but its components are described in the fixed observing frame. The vectors $\vec{z}$ and $\vec{r}$ are two-dimensional for planar rigid bodies, and three-dimensional for general spatial rigid bodies. R is a rotation matrix, and θ parametrizes a rigid body rotation. In the case of planar rigid bodies, the rotation matrix takes the simple form:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} .$$

Velocity Transformations: If the rigid body moves with respect to the fixed observing frame, then the velocity of the point P on the rigid body, as seen by the observer in the fixed frame, is:

$$\vec{V} = \frac{d\vec{z}}{dt} = \dot{\vec{d}} + R(\theta)(\vec{\omega} \times \vec{r}) \tag{2}$$

where $\vec{\omega}$ is the *angular velocity* of the moving rigid body, and $\dot{\vec{d}}$ describes the velocity of the point in the moving body which is coincident with body fixed reference frame.

2 Kinematic Analysis of the Vehicle

The goal of this section is to derive a relationship between the wheel speeds and the vehicle speed. In order to derive this relationship, we will make the following assumptions:

A1: each wheel contacts the ground at a single point,

A2: both wheels roll on the ground without slipping.

If the wheels roll without slipping, then the point on each wheel (whose Cartesian location is denoted by $\vec{C}_1$ for the 1^{st} or right wheel, and by $\vec{C}_2$ for the 2^{nd} or left wheel) which is instantaneously in contact with the ground must have zero velocity. Else, if that point has non-zero velocity, the wheel must be slipping with respect to the ground–in this case we need to implement a more complicated analysis.

2.1 Velocities of the wheel-ground contact points

To calculate the velocity of points $\vec{C}_1$ and $\vec{C}_2$ (denoted $\vec{V}_{C_1}$ and $\vec{V}_{C_2}$), we will resort to "diagram chasing." Diagram chasing involves a sequence of simple rigid body transformations of the types summarized in the last section.

Let $\vec{H}_1$ and $\vec{H}_2$ respectively denote the location of the "hubs" on both wheels. Idealized, the hub is the point that defines the center of the wheel's rotation. This point is rigidly affixed to the main vehicle body, and thus the velocity of the hub points (denoted $\vec{V}_{H_1}$ and $\vec{V}_{H_2}$) can be calculated from the vehicle's velocity via application of formula (2). Because $\vec{C}_1$ is a point on the rotating rigid body wheel, its velocity can also be calculated from Eq. (2) because we know the speed of the hub, and we assume that we know the rotational speed of the wheel (which defines the relative motion of the wheel with respect to the hub).

The velocity of hub 1 is:

$$\vec{V}_{H_1} = \vec{V}_R + \vec{\omega}_R \times R\vec{r}_{H_1} \tag{3}$$

where $\vec{V}_R$ and $\vec{\omega}_R$ are the linear and angular velocities of the robot:

$$\vec{V}_R = \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} \qquad \vec{\omega}_R = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \tag{4}$$

and $\vec{r}_{H_1}$ is the vector from origin of the body fixed frame to the hub point, and R denotes the relative orientation of the body fixed reference frame with respect to the fixed observing reference frame:

$$\vec{r}_{H_1} = \begin{bmatrix} 0 \\ -W \\ 0 \end{bmatrix} \qquad R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

Substituting Eq.s (4) and (5) into Eq. (3) yields:

$$\vec{V}_{H_1} = \begin{bmatrix} \dot{x} + W\dot{\theta}\cos\theta \\ \dot{y} + W\dot{\theta}\sin\theta \\ 0 \end{bmatrix} \tag{6}$$

Using a similar analysis, the velocity of the second hub is:

$$\vec{V}_{H_2} = \begin{bmatrix} \dot{x} - W\dot{\theta}\cos\theta \\ \dot{y} - W\dot{\theta}\sin\theta \\ 0 \end{bmatrix} \tag{7}$$

Since the point $\vec{C}_1$ is rigidly affixed to moving the wheel #1, and we know the velocity of the hub, $\vec{V}_{H_1}$, then:

$$\vec{V}_{C_1} = \vec{V}_{H_1} + \vec{\omega}_{W_1} \times \vec{r}_{H_1C_2} \tag{8}$$

where $\vec{\omega}_{W_1}$ is the angular velocity of wheel 1 and $\vec{r}_{H_1C_1}$ is the vector from the point of hub 1 to $\vec{C}_1$:

$$\vec{\omega}_{W_1} = \dot{\phi}_1 \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix} \qquad \vec{r}_{H_1C_2} = \begin{bmatrix} 0 \\ 0 \\ \rho \end{bmatrix} . \tag{9}$$

Substituting Eq.s (9) and (6) into Eq. (8) yields:

$$\vec{V}_{C_1} = \begin{bmatrix} \dot{x} + W\dot{\theta}\cos\theta + \rho\dot{\phi}_1\cos\theta \\ \dot{y} + W\dot{\theta}\sin\theta + \rho\dot{\phi}_1\cos\theta \\ 0 \end{bmatrix} \tag{10}$$

Using a similar derivation, the velocity of the contact point on the other wheel can be found as:

$$\vec{V}_{C_2} = \vec{V}_{H_2} + \vec{\omega}_{W_2} \times \vec{r}_{H_2C_2} = \begin{bmatrix} \dot{x} - W\dot{\theta}\cos\theta - \rho\dot{\phi}_2\cos\theta \\ \dot{y} - W\dot{\theta}\sin\theta - \rho\dot{\phi}_2\sin\theta \\ 0 \end{bmatrix} \tag{11}$$

2.2 Using the Nonholonomic Constraints

The constraint that the velocities of the points on the wheel in contact with the ground are zero yields these four equations:

$$\dot{x} + W\dot{\theta}\cos\theta + \rho\dot{\phi}_1\cos\theta = 0 \tag{12}$$
$$\dot{y} + W\dot{\theta}\sin\theta + \rho\dot{\phi}_1\sin\theta = 0 \tag{13}$$
$$\dot{x} - W\dot{\theta}\cos\theta - \rho\dot{\phi}_2\cos\theta = 0 \tag{14}$$
$$\dot{y} - W\dot{\theta}\sin\theta - \rho\dot{\phi}_2\sin\theta = 0\ . \tag{15}$$

These equations give us four equations in the three unknowns $(\dot{x}, \dot{y}, \dot{\theta})$, assuming that $\dot{\phi}_1$ $\dot{\phi}_2$ are known. However, one of these equations is redundant, and thus we have 3 unique equations in 3 unknowns. Thus, we can solve for the vehicle's velocity as a function of the wheel velocities..

To realize the desired relationship, Equations (12)-(15) can be manipulated in the following fashion. Add Equations (12) and (14)

$$2\dot{x} - \rho(\dot{\phi}_2 - \dot{\phi}_1)\cos\theta = 0 \ \Rightarrow\ \dot{x} = \frac{\rho}{2}(\dot{\phi}_2 - \dot{\phi}_1)\cos\theta\ . \tag{16}$$

Similarly, add Equations (13) and (15)

$$2\dot{y} - \rho(\dot{\phi}_2 - \dot{\phi}_1)\sin\theta = 0 \ \Rightarrow\ \dot{y} = \frac{\rho}{2}(\dot{\phi}_2 - \dot{\phi}_1)\sin\theta\ . \tag{17}$$

Next, subtract Equation (14) from (12) and subtract Eq. (15) from Eq. (13) to yield these two equations:

$$2W\dot{\theta}\cos\theta + \rho(\dot{\phi}_1 + \dot{\phi}_2)\cos\theta = 0 \tag{18}$$
$$2W\dot{\theta}\sin\theta + \rho(\dot{\phi}_1 + \dot{\phi}_2)\sin\theta = 0\ . \tag{19}$$
$$\tag{20}$$

Multiply Eq. (18) by $\cos\theta$ and multiply Eq. (19) by $\sin\theta$. Add the two resulting equations to yield:

$$2W\dot{\theta} + \rho(\dot{\phi}_1 + \dot{\phi}_2) = 0 \quad\Rightarrow\quad \dot{\theta} = -\frac{\rho}{2W}(\dot{\phi}_1 + \dot{\phi}_2)\ . \tag{21}$$

In summary, the wheel motions $\dot{\phi}_1$ and $\dot{\phi}_2$ are related to the vehicle motion as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \frac{\rho}{2}\begin{bmatrix} (\dot{\phi}_2 - \dot{\phi}_1)\cos\theta \\ (\dot{\phi}_2 - \dot{\phi}_1)\sin\theta \\ -(\dot{\phi}_2 + \dot{\phi}_1)/W \end{bmatrix} \tag{22}$$

which can be equivalently expressed as:

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \frac{\rho}{2} \begin{bmatrix} (\dot{\phi}_2 - \dot{\phi}_1) \\ (\dot{\phi}_2 - \dot{\phi}_1) \\ -(\dot{\phi}_2 + \dot{\phi}_1)/W \end{bmatrix} . \tag{23}$$

2.3 Conclusions from Kinematic Analysis

If the goal is to design the vehicle for the maximum top end speed, then Equations (23) suggest that:

- maximize the top end wheel speed (this is done by selecting the right gear ratio).
- maximize the wheel diameter, ρ.
- minimize the wheel base, W.