ME 72: Engineering Design Laboratory
Analysis of a Differential Drive Vehicle

1 Introduction

These notes are a companion to the in-class lectures on the *differential drive* ground vehicle. Figure 1 shows both a top view and a side view of an idealized differential drive vehicle.

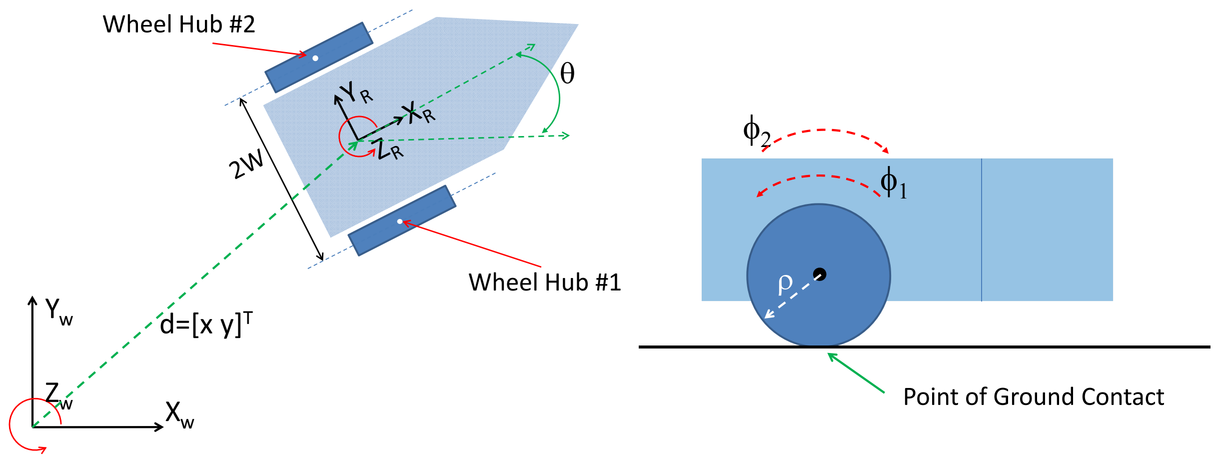

Figure 1: Schematic diagram of an idealized differential drive vehicle. **Left:** top view of the vehicle. **Right:** Side view of the vehicle.

It is very likely that many teams will build differential drive (or similar) ground vehicles, and therefore a basic understand of their kinematics and dynamics may be useful to those teams. The goal of these notes is to derive *parametric models* that:

- relate the wheel speed to vehicle speed (as a function of key parameters).
- relate wheel torques to vehicle accelerations (as a function of key parameters).

The notes will also attempt some preliminary analysis of how key parameter choices may affect vehicle performance. However, additional analysis beyond these notes should be pursued by those teams looking to optimize their vehicle designs.

1.1 Basic Kinematic Principles

To derive our results, we will use some very basic principles of rigid body kinematics. Consider a moving rigid body (see Figure 2). Attach a reference frame, $\mathcal{F}_2$ (with unit basis vectors $\{x_2, y_2, z_2\}$), to the moving body. Assume that there is an additional, fixed observing frame $\mathcal{F}_1$ (whose unit basis vectors are $\{x_1, y_1, z_1\}$). We will use the basic notions of coordinate transformations between points in the moving and fixed reference frames, as well as the notion of the velocity of points in the moving body.

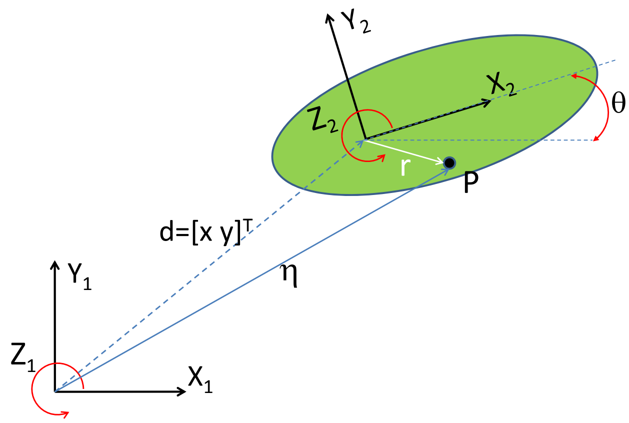

Figure 2: Geometry of a moving rigid body

Coordinate Transformations: The transformation between the coordinates of a point P in the moving rigid body (as described by an observer in $\mathcal{F}_2$) to its equivalent representation in the fixed observing reference frame $\mathcal{F}_1$ is:

$$\vec{\eta} = \vec{d} + R(\theta)\vec{r} \tag{1}$$

where $\vec{d}$ is the vector from the origin of $\mathcal{F}_1$ to the origin of the $\mathcal{F}_2$, and where $\vec{r}$ is the vector to point P, as described in reference frame $\mathcal{F}_2$ (see Figure 2). The vector $\vec{\eta}$ points from the origin of $\mathcal{F}_1$ frame to that same point P, but its components are described in the fixed observing frame $\mathcal{F}_1$. The vectors $\vec{\eta}$ and $\vec{r}$ are two-dimensional for planar rigid bodies, and three-dimensional for general spatial rigid bodies. R is a rotation matrix, and θ parametrizes a rigid body rotation. In the case of planar rigid bodies, the rotation matrix takes the simple form:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} .$$

Velocity Transformations: If the rigid body moves with respect to the fixed observing frame, then the velocity of the point P on the rigid body, as seen by the observer in the fixed frame, $\mathcal{F}_2$, is:

$$\vec{V} = \frac{d\vec{\eta}}{dt} = \dot{\vec{d}} + \vec{\omega}(R(\theta) \times \vec{r}) \tag{2}$$

where $\vec{\omega}$ is the *spatial angular velocity* of the moving rigid body (the rate of rotation and the rotation axis, as described in frame $\mathcal{F}_1$), and $\dot{\vec{d}}$ describes the velocity of the point in the moving body which is coincident with the origin of frame $\mathcal{F}_2$. One can think of $\dot{\vec{d}}$ as the *translational velocity* of the vehicle.

2 Kinematic Analysis of the Vehicle

The goal of this section is to derive a relationship between the wheel speeds and the vehicle speed. In order to derive this relationship, we will make the following assumptions:

A1: each wheel contacts the ground at a single point,

A2: both wheels roll on the ground without slipping.

If the wheels roll without slipping, then the point on each wheel (whose Cartesian location is denoted by $\vec{C}_1$ for the 1^{st} (right) wheel, and by $\vec{C}_2$ for the 2^{nd} (left) wheel) which is instantaneously in contact with the ground must have zero velocity. Else, if that point has non-zero velocity, the wheel must be slipping with respect to the ground–in this case we need to implement a more complicated analysis.

2.1 Velocities of the wheel-ground contact points

To calculate the velocity of points $\vec{C}_1$ and $\vec{C}_2$ (denoted $\vec{V}_{C_1}$ and $\vec{V}_{C_2}$), we will resort to "diagram chasing." Diagram chasing involves a sequence of simple rigid body transformations of the types summarized in the last section.

Let $\vec{H}_1$ and $\vec{H}_2$ respectively denote the location of the "hubs" on both wheels (see Figure 1). Idealized, the hub is the point that defines the center of the wheel's rotation. This point is rigidly affixed to the main vehicle body, and thus the velocity of the hub points (denoted $\vec{V}_{H_1}$ and $\vec{V}_{H_2}$) can be calculated from the vehicle's velocity via application of formula (2). Because $\vec{C}_1$ is a point on the rotating rigid body wheel, its velocity can also be calculated from Eq. (2) because we know the speed of the hub, and we assume that we know the rotational speed of the wheel (which defines the relative motion of the wheel with respect to the hub).

The velocity of hub 1 is:

$$\vec{V}_{H_1} = \vec{V}_R + \vec{\omega}_R \times R\vec{r}_{H_1} \tag{3}$$

where $\vec{V}_R$ and $\vec{\omega}_R$ are the linear and angular velocities of the robot:

$$\vec{V}_R = \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} \qquad \vec{\omega}_R = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \tag{4}$$

and $\vec{r}_{H_1}$ is the vector from origin of the body fixed frame to the hub point, and R denotes the relative orientation of the body fixed reference frame with respect to the fixed observing reference frame:

$$\vec{r}_{H_1} = \begin{bmatrix} 0 \\ -W \\ 0 \end{bmatrix} \qquad R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

where $2W$ is half of the width of the wheel base. Substituting Eq.s (4) and (5) into Eq. (3) yields:

$$\vec{V}_{H_1} = \begin{bmatrix} \dot{x} + W\dot{\theta}\cos\theta \\ \dot{y} + W\dot{\theta}\sin\theta \\ 0 \end{bmatrix} \tag{6}$$

Using a similar analysis, the velocity of the second hub is:

$$\vec{V}_{H_2} = \begin{bmatrix} \dot{x} - W\dot{\theta}\cos\theta \\ \dot{y} - W\dot{\theta}\sin\theta \\ 0 \end{bmatrix} \tag{7}$$

Since the point $\vec{C}_1$ is rigidly affixed to moving the wheel #1, and we know the velocity of the hub, $\vec{V}_{H_1}$, then:

$$\vec{V}_{C_1} = \vec{V}_{H_1} + \vec{\omega}_{W_1} \times \vec{r}_{H_1C_2} \tag{8}$$

where $\vec{\omega}_{w_1}$ is the angular velocity of wheel 1 and $\vec{r}_{H_1C_1}$ is the vector from the point of hub 1 to $\vec{C}_1$:

$$\vec{\omega}_{W_1} = \dot{\phi}_1 \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix} \qquad \vec{r}_{H_1C_2} = \begin{bmatrix} 0 \\ 0 \\ \rho \end{bmatrix} . \tag{9}$$

Substituting Eq.s (9) and (6) into Eq. (8) yields:

$$\vec{V}_{C_1} = \begin{bmatrix} \dot{x} + W\dot{\theta}\cos\theta + \rho\dot{\phi}_1\cos\theta \\ \dot{y} + W\dot{\theta}\sin\theta + \rho\dot{\phi}_1\cos\theta \\ 0 \end{bmatrix} \tag{10}$$

Using a similar derivation, the velocity of the contact point on the other wheel can be found as:

$$\vec{V}_{C_2} = \vec{V}_{H_2} + \vec{\omega}_{W_2} \times \vec{r}_{H_2C_2} = \begin{bmatrix} \dot{x} - W\dot{\theta}\cos\theta - \rho\dot{\phi}_2\cos\theta \\ \dot{y} - W\dot{\theta}\sin\theta - \rho\dot{\phi}_2\sin\theta \\ 0 \end{bmatrix} \tag{11}$$

2.2 Using the Nonholonomic Constraints

The constraint that the points on the wheel in contact with the ground must have zero velcoity results in these four equations:

$$\begin{aligned} \dot{x} + W\dot{\theta}\cos\theta + \rho\dot{\phi}_1\cos\theta &= 0 && (12) \\ \dot{y} + W\dot{\theta}\sin\theta + \rho\dot{\phi}_1\sin\theta &= 0 && (13) \\ \dot{x} - W\dot{\theta}\cos\theta - \rho\dot{\phi}_2\cos\theta &= 0 && (14) \\ \dot{y} - W\dot{\theta}\sin\theta - \rho\dot{\phi}_2\sin\theta &= 0 . && (15) \end{aligned}$$

These equations give us four equations in the three unknowns $(\dot{x}, \dot{y}, \dot{\theta})$, assuming that $\dot{\phi}_1$ $\dot{\phi}_2$ are known. However, one of these equations is redundant, and thus we have 3 unique equations in 3 unknowns. Thus, we can solve for the vehicle's velocity as a function of the wheel velocities.

To realize the desired relationship, Equations (12)-(15) can be manipulated in the following fashion. Add Equations (12) and (14)

$$2\dot{x} - \rho(\dot{\phi}_2 - \dot{\phi}_1)\cos\theta = 0 \;\Rightarrow\; \dot{x} = \frac{\rho}{2}(\dot{\phi}_2 - \dot{\phi}_1)\cos\theta \;. \tag{16}$$

Similarly, add Equations (13) and (15)

$$2\dot{y} - \rho(\dot{\phi}_2 - \dot{\phi}_1)\sin\theta = 0 \;\Rightarrow\; \dot{y} = \frac{\rho}{2}(\dot{\phi}_2 - \dot{\phi}_1)\sin\theta \;. \tag{17}$$

Next, subtract Equation (14) from (12) and subtract Eq. (15) from Eq. (13) to yield these two equations:

$$2W\dot{\theta}\cos\theta + \rho(\dot{\phi}_1 + \dot{\phi}_2)\cos\theta = 0 \tag{18}$$

$$2W\dot{\theta}\sin\theta + \rho(\dot{\phi}_1 + \dot{\phi}_2)\sin\theta = 0 \;. \tag{19}$$

Multiply Eq. (18) by $\cos\theta$ and multiply Eq. (19) by $\sin\theta$. Add the two resulting equations to yield:

$$2W\dot{\theta} + \rho(\dot{\phi}_1 + \dot{\phi}_2) = 0 \quad\Rightarrow\quad \dot{\theta} = -\frac{\rho}{2W}(\dot{\phi}_1 + \dot{\phi}_2) \;. \tag{20}$$

In summary, the wheel motions $\dot{\phi}_1$ and $\dot{\phi}_2$ are related to the vehicle motion as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \frac{\rho}{2}\begin{bmatrix} (\dot{\phi}_2 - \dot{\phi}_1)\cos\theta \\ (\dot{\phi}_2 - \dot{\phi}_1)\sin\theta \\ -(\dot{\phi}_2 + \dot{\phi}_1)/W \end{bmatrix} \tag{21}$$

which can be equivalently expressed as:

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \frac{\rho}{2}\begin{bmatrix} (\dot{\phi}_2 - \dot{\phi}_1) \\ (\dot{\phi}_2 - \dot{\phi}_1) \\ -(\dot{\phi}_2 + \dot{\phi}_1)/W \end{bmatrix} \;. \tag{22}$$

2.3 Conclusions from Kinematic Analysis

If the goal is to design the vehicle for the maximum top end speed, then from a purely kinematic point of view, Equations (22) suggest that to produce a vehicle with the best top end speed (to traverse the contest course as quickly as possible).

- maximize the top end wheel speed (this is done by selecting the right gear ratio).
- maximize the wheel diameter, ρ.
- minimize the wheel base, W.

3 Dynamic Analysis of the Differential Drive Vehicle

Next we need to move beyond a purely kinematic analysis to consider how dynamic variables (the magnitudes of masses and inertias), as well as the geometric parameters, affect vehicle performance. In this section our goal is to develop expressions that relate the vehicle's acceleration to the motor torques and other design parameters. To do this, we must develop the equations that govern the vehicle's motion. We can either take a Lagrangian or Newtonian approach. These notes pursue a Lagrangian method, but the same results can be derived either way.

3.1 Lagrange's Equations of Motion

Recall that the Lagrangian approach to deriving the equations of motion is based on the definition of a *Lagrangian*:

$$\mathcal{L} \;=\; K.E. \;-\; P.E. \;=\; \mathcal{L}(q, \dot{q})$$

where $K.E.$ is the total kinetic energy of the dynamic sytem andd P.E. is its potential energy. The variable q denotes the *configuration* of the system, which is the set of variables that can uniquely describe the position of the system at a fixed time. For the differential drive vehicle, the system configuration consists of the 5-tuple

$$\vec{q} = \begin{bmatrix} x & y & \theta & \phi_1 & \phi_2 \end{bmatrix}^T \ .$$

For an unconstrained system, Lagrange's equations of motion are derived from the Lagrangian as follows:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(\vec{q}, \dot{\vec{q}})}{\partial \dot{\vec{q}}}\right] - \frac{\partial \mathcal{L}(\vec{q}, \dot{\vec{q}})}{\partial \vec{q}} \;=\; \vec{T} \tag{23}$$

where $\vec{T}$ are the *generalized forces* acting on the system (see below). Howver, when constraints are at work, the Lagrangian approach must be adapted to include the influence of the constraints. This is formally done using the *method of undertermined Lagrange Multipliers.*

The no-slip wheel constraints of Eq. (22) (formally called *nonholonomic constraints*) developed in the last section can be rearranged to in the form:

$$C(q)\dot{\vec{q}} \;=\; 0$$

where for the differential drive vehicle, $C(\vec{q})$ is the 3×5 matrix

$$C(q) = \begin{bmatrix} \cos\theta & \sin\theta & 0 & \frac{\rho}{2} & -\frac{\rho}{2} \\ -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{\rho}{2W} & \frac{\rho}{2W} \end{bmatrix} \ . \tag{24}$$

When nonholonomic constraints of the form $C(\vec{q})\dot{\vec{q}} = 0$ are present, Lagrange's equations take the form:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(\vec{q},\dot{\vec{q}})}{\partial \dot{\vec{q}}}\right] - \frac{\partial \mathcal{L}(\vec{q},\dot{\vec{q}})}{\partial \vec{q}} - C(\vec{q})^T\vec{\lambda} = \vec{T} \tag{25}$$

where $\vec{\lambda}$ is the vector of *undetermined Lagrange multipliers*[1]. The multipliers represent the forces which are necessary to maintain the constraints. For the differential drive vehicle, these multipliers are the reaction forces at the wheel-ground contact which are necessary to enforce the no-slip wheel constraints.

Let us first consider the case where the vehicle rolls around on a flat floor (i.e., we'll ignore the case where the vehicle is climbing up the ramp). In this case, the gravitational potential energy is constant, and so the Lagrangian is comprised solely of kinetic energy terms:

$$\mathcal{L}(\vec{q},\dot{\vec{q}}) = K.E._{total} = \sum_{i=1}^{N_{body}} K.E._i$$

where N_{body} is the number of rigid bodies in the overall system, and $K.E._i$ is the kinetic energy of the i^{th} body. The differential drive vehicle is made up of three bodies: (1) the main body of the vehicle; (2) the right wheel; (3) the left wheel (other internal moving parts are ignored in this analysis, but might be important for your specific contest design). The kinetic energy of the i^{th} rigid body is:

$$K.E. = \frac{1}{2}m_i||\vec{V}_i||^2 + \frac{1}{2}\Omega_i^T I_i \Omega_i \tag{26}$$

where

$$\begin{aligned} m_i &= \text{the mass of the i}^{\text{th}}\text{ body} \\ \vec{V}_i &= \text{the velocity of the i}^{\text{th}}\text{ body's center of mass} \\ \Omega_i &= \text{the angular velocity of the i}^{\text{th}}\text{ body} \\ I_i &= \text{the inertia tensor of the i}^{\text{th}}\text{ body about its center of mass .} \end{aligned}$$

3.2 Kinetic Energy for the Differential Drive System

This section will develop each of the components of the overall system kinetic energy: the kinetic energy of the main body, and the kinetic energy of the two wheels.

Kinetic Energy of Main Body: Let us assume that the center of mass of the main vehicle body is located at a distance d along the x-axis of the body fixed reference frame. Then the

[1]From the principle of *virtual work*, the constraint forces should do no work on the system. Note that if $C(\vec{q})^T\vec{\lambda}$ represent these constraint forces, then the work done by the constraint forces is: $\left(C(\vec{q})^T\vec{\lambda}\right)\cdot\dot{\vec{q}} = \left(C(\vec{q})^T\vec{\lambda}\right)^T\dot{\vec{q}} = \vec{\lambda}^T C(\vec{q})\dot{\vec{q}} = 0$.

velocity of the main body's center of mass, $\vec{V}_B$ is:

$$\vec{V}_B = \vec{V}_R + \vec{\omega}_R \times \vec{r}_{CM_1} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \times \begin{bmatrix} d\cos\theta \\ d\sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{x} - d\dot{\theta}\sin\theta \\ \dot{y} + d\dot{\theta}\cos\theta \\ 0 \end{bmatrix} \tag{27}$$

where $\vec{V}_R$ is the velocity of the robot (i.e., the velocity of the origin of the body fixed reference frame). Hence, the total kinetic energy of the main body is:

$$\begin{aligned} K.E._{mainbody} &= m_B||\vec{V}_1||^2 + \vec{\omega}_B^T I_B \vec{\omega}_B \\ &= \frac{m_B}{2}(\dot{x}^2 + \dot{y}^2 + d^2\dot{\theta}^2 + 2d\dot{\theta}(\dot{y}\cos\theta - \dot{x}\sin\theta)) + \frac{I_B}{2}\dot{\theta}^2 \end{aligned} \tag{28}$$

where m_B is the mass of the main body, I_B is the rotational inertia of the main body about the axis normal to the plane of its movement, and $\vec{\omega}_B$ is the angular velocity of the main body, which is simply $\vec{\omega}_B = \vec{\omega}_R = \begin{bmatrix} 0 & 0 & \dot{\theta} \end{bmatrix}^T$.

Kinetic Energy of the Wheels. Let us analyze the kinetic energy of the right wheel (the second body). The kinetic energy of the second wheel (third body) can be computed in a highly analogous fashion.

Assume that the wheel's center of mass lies exactly at the "hub" point that was previously analyzed in Section 2. Hence, the velocity of the right wheel's center of mass is:

$$\vec{V}_{CM_1} = \vec{V}_{H_1} = \begin{bmatrix} \dot{x} + W\dot{\theta}\cos\theta \\ \dot{y} + W\dot{\theta}\sin\theta \\ 0 \end{bmatrix} \tag{29}$$

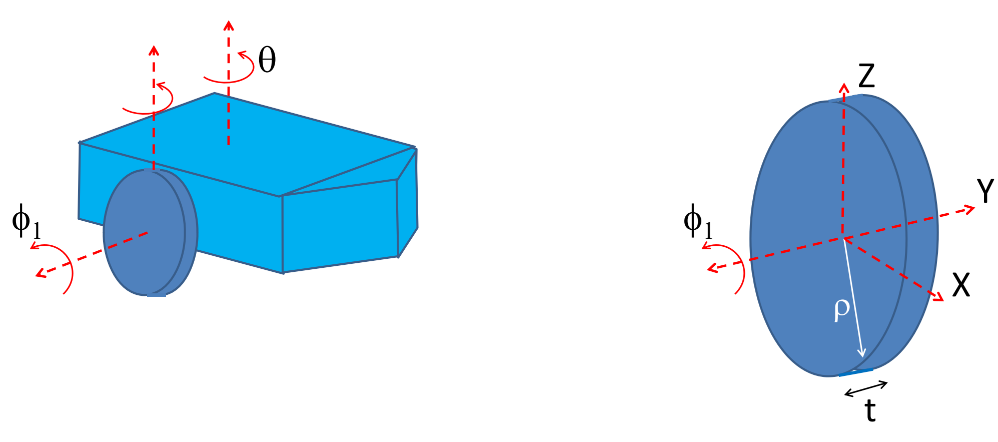

Figure 3: **(a):** Perspective view of vehicle, stressing that wheels also rotate about a vertical axis as the entire vehicle rotates about the axis normal to the plane; **(b):** Geometry of a cylindrical wheel with thickness t and radius ρ.

The right wheel not only rotates about its axle, but it also rotates about an axis normal to the plane (in concert with the rotation of the entire vehicle about the vertical (see Figure 3(a)). Hence, it's angular velocity, $\vec{\Omega}_{w_1}$, is:

$$\vec{\Omega}_{w_1} = \dot{\phi}_1 \begin{bmatrix} \sin\theta \\ \cos\theta \\ 0 \end{bmatrix} + \dot{\theta} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \dot{\phi}_1 \sin\theta \\ \dot{\phi}_1 \cos\theta \\ \dot{\theta} \end{bmatrix} .$$

Let us assume that the wheel is a homogeneous cylinder (Figure 3(b)). Recall that the inertia tensor of a such a cylinder (using the coordinate system shown in Figure 3(b)) is:

$$I_w^b = \begin{bmatrix} I_{xx}^b & 0 & 0 \\ 0 & I_{yy}^b & 0 \\ 0 & 0 & I_{zz}^b \end{bmatrix} = \begin{bmatrix} \frac{m_w}{12}(3\rho^2 + t^2) & 0 & 0 \\ 0 & \frac{m_w \rho^2}{2} & 0 \\ 0 & 0 & \frac{m_w}{12}(3\rho^2 + t^2) \end{bmatrix} \tag{30}$$

where m_w is the mass of the wheel, t is the thickness of the wheel, and the notation I^b indicates that the inertia tensor is described in the body fixed reference frame of Figure 3(b). If we assume that the wheel is "thin," then

$$I_{xx}^b = I_{zz}^b \simeq \frac{m_w \rho^2}{4} .$$

Note that the general notation for the components of the wheel inertial will be kept for now, and the specific formula for the inertial terms will be substituted later when appropriate. Note also that we must tranform the inertia tensor, which is described in Eq. (30) with respect to the body fixed frame of Figure 3(b), to the coordinate system of the observing reference frame. This coordinate transformation is a simple rotation about the vertical axis by angle θ:

$$I_w = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{xx}^b & 0 & 0 \\ 0 & I_{yy}^b & 0 \\ 0 & 0 & I_{zz}^b \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Integrating the above developments and formulas, the kinetic energy of the wheel is:

$$\begin{aligned} K.E._{wheel_1} &= \frac{m_w}{2}||V_{CM_1}||^2 + \frac{1}{2}\vec{\Omega}_{w_1}^T I_w \vec{\Omega}_{w_1} \\ &= \frac{m_w}{2}(\dot{x}^2 + \dot{y}^2 + W^2\dot{\theta}^2 + 2W\dot{\theta}(\dot{x}\cos\theta + \dot{y}\sin\theta)) + \frac{I_{zz}^b}{2}\dot{\theta}^2 + \frac{I_{yy}^b}{2}\dot{\phi}_1^2 \end{aligned} \tag{31}$$

An analogous derivation for the left (second) wheel yields:

$$\begin{aligned} K.E._{wheel_2} &= \frac{m_w}{2}||V_{CM_2}||^2 + \frac{1}{2}\vec{\Omega}_{w_2}^T I_w \vec{\Omega}_{w_2} \\ &= \frac{m_w}{2}(\dot{x}^2 + \dot{y}^2 + W^2\dot{\theta}^2 - 2W\dot{\theta}(\dot{x}\cos\theta + \dot{y}\sin\theta)) + \frac{I_{zz}^b}{2}\dot{\theta}^2 + \frac{I_{yy}^b}{2}\dot{\phi}_2^2 \end{aligned} \tag{32}$$

Total System Kinetic Energy: The total kinetic energy is simply the sum of the kinetic energies of each body. Substituting the expressions derived above and then rearranging produces the desired kinetic energy expression (which also equals the system Lagrangian under the assumption that the vehicle drives on the flat gymnasium floor):

$$\begin{aligned} K.E._{total} &= K.E._{main\ body} + K.E._{wheel_1} + K.E._{wheel_2} \\ &= \frac{m_B}{2}\left(\dot{x}^2 + \dot{y}^2 + d^2\dot{\theta}^2 + 2d\dot{\theta}(\dot{y}\cos\theta - \dot{x}\sin\theta)\right) + \frac{I_B}{2}\dot{\theta}^2 + \frac{m_w}{2}(\dot{x}^2 + \dot{y}^2 + W^2\dot{\theta}^2) \\ &\quad + \frac{I_{yy}^b}{2}\dot{\phi}_1^2 + \frac{I_{zz}^b}{2}\dot{\theta}^2 + \frac{m_w}{2}(\dot{x}^2 + \dot{y}^2 + W^2\dot{\theta}^2) + \frac{I_{yy}^b}{2}\dot{\phi}_2^2 + \frac{I_{zz}^b}{2}\dot{\theta}^2 \\ &= \frac{1}{2}(m_B + 2m_w)(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(m_B d^2 + I_B + 2m_w W^2 + 2I_{zz}^b)\dot{\theta}^2 \\ &\quad + m_B d\dot{\theta}(\dot{y}\cos\theta - \dot{x}\sin\theta) + \frac{I_{yy}^b}{2}(\dot{\phi}_1^2 + \dot{\phi}_2^2) \end{aligned} \tag{33}$$

Introducing the variables

$$m_T = m_B + 2m_w = \text{total vehicle mass} \tag{34}$$

$$I_T = I_B + m_B d^2 + 2m_w W^2 + 2I_{zz}^b = \text{total rotational inertial} \tag{35}$$

the total kinetic energy expression simplifies to:

$$K.E._{total} = \mathcal{L}(\vec{q}, \dot{\vec{q}}) = \frac{m_T}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I_T}{2}\dot{\theta}^2 + m_B d\dot{\theta}(\dot{y}\cos\theta - \dot{x}\sin\theta) + \frac{I_{yy}^b}{2}(\dot{\phi}_1^2 + \dot{\phi}_2^2) \tag{36}$$

3.3 Lagrange's Equations for the Differential Drive Vehicle

Substituting Equations (36 and (24) into Lagranges's equations

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}\right] - \frac{\partial \mathcal{L}(\vec{q}, \dot{\vec{q}})}{\partial \vec{q}} - C(\vec{q})^T \vec{\lambda} = \vec{T}$$

results in

$$\frac{d}{dt}\begin{bmatrix} m_T\dot{x} - m_B d\dot{\theta}\sin\theta \\ m_T\dot{y} + m_B d\dot{\theta}\cos\theta \\ I_T\dot{\theta} + m_B d(\dot{y}\cos\theta - \dot{x}\sin\theta) \\ I_{yy}^b\dot{\phi}_1 \\ I_{yy}^b\dot{\phi}_1 \end{bmatrix} + m_B d\dot{\theta}\begin{bmatrix} 0 \\ 0 \\ \dot{y}\sin\theta + \dot{x}\cos\theta \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \frac{\rho}{2} & 0 & \frac{\rho}{2W} \\ -\frac{\rho}{2} & 0 & \frac{\rho}{2W} \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tau_1 \\ \tau_2 \end{bmatrix}$$

where τ_1 and τ_2 are the torques applied to the wheels. Taking derivatives and rearranging produces an equation of the form:

$$M(q)\ddot{\vec{q}} + B(\vec{q}, \dot{\vec{q}}) - C^T(\vec{q})\vec{\lambda} = \vec{T} \tag{37}$$

where:

$$M(\vec{q}) = \begin{bmatrix} m_T & 0 & -m_B d\sin\theta & 0 & \\ 0 & m_T & m_B d\cos\theta & 0 & \\ -m_B d\sin\theta & m_B d\cos\theta & I_T & 0 & \\ 0 & 0 & 0 & I_{yy}^b & 0 \\ 0 & 0 & 0 & 0 & I_{yy}^b \end{bmatrix}; \quad B(\vec{q}, \dot{\vec{q}}) = -m_B d\dot{\theta}^2\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{38}$$

$C(\vec{q})$ was defined above, and $\vec{T} = \begin{bmatrix} 0 & 0 & 0 & \tau_1 & \tau_2 \end{bmatrix}^T$.

3.4 Finding the Lagrange Multipliers

While Equation (37) provides a complete dynamical characterization of the differential drive vehicle, we must still find the undetermined Lagrange multipliers, $\vec{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}^T$. A general procedure to find the multipliers for equations of the form Eq. (37) follows.

We know that the holonomic constraints take the form $C(\vec{q})\dot{\vec{q}} = 0$. Hence, it must be true that:

$$\frac{d}{dt}\left[C(\vec{q})\dot{\vec{q}}\right] = 0 \qquad \Rightarrow \qquad C(\vec{q})\ddot{\vec{q}} + \dot{C}(\vec{q})\dot{\vec{q}} = 0 \tag{39}$$

Solving Eq. (37) for $\ddot{\vec{q}}$

$$\ddot{\vec{q}} = M^{-1}(\vec{q})\left[\vec{T} - B(\vec{q},\dot{\vec{q}}) + C(\vec{q})^T\vec{\lambda}\right]$$

which can then be substituted into Eq. (39) to yield:

$$\left[C(\vec{q})M^{-1}(\vec{q})C^T(\vec{q})\right]\vec{\lambda} = -\left[C(\vec{q})M^{-1}(\vec{q})[\vec{T} - B(\vec{q},\dot{\vec{q}})] + \dot{C}(\vec{q})\dot{\vec{q}}\right] . \tag{40}$$

If the constraint equations are well defined, then the matrix $\left[C(\vec{q})M^{-1}(\vec{q})C^T(\vec{q})\right]$ is invertible, allowing for $\vec{\lambda}$ to be calculated as:

$$\vec{\lambda} = -\left[C(\vec{q})M^{-1}(\vec{q})C^T(\vec{q})\right]^{-1}\left[C(\vec{q})M^{-1}(\vec{q})(\vec{T} - B(\vec{q},\dot{\vec{q}})) + \dot{C}(\vec{q})\dot{\vec{q}}\right] . \tag{41}$$

Thus, in principle one can solve for the undetermined multipliers λ, and then subtitute them back into Lagrange's equations (37) to produce practical formulas that relate system accelerations to wheel torques. However, even for the relatively simple differential drive vehicle, this process will produce a messy outcome.

3.5 Analysis of the Dynamical Equations

Let's introduce a few simplifying assumptions that will produce tractable equations which still capture the essential features of the problem.

Assumption 1: Let $d = 0$ (i.e., the main body's center of mass lies along the wheel axles). With this simplification, $B(\vec{q},\dot{\vec{q}}) = 0$, and $M(q)$ becomes a diagonal matrix:

$$M(q) = diag(m_T, m_T, I_T, I_{yy}^2, I_{yy}^2) .$$

Assumption 2: Let's evaluate the dynamical equations at $\theta = 0$. This is not a simplification, but instead represents a special choice of the observing coordinate system to be parallel to the body-fixed coordinate system. This choice will eliminate some terms in the equations of motion.

With these simplifications

$$\vec{\lambda} = -\left[C(\vec{q})M^{-1}(\vec{q})C^T(\vec{q})\right]^{-1}\left(C(\vec{q})M^{-1}(\vec{q})\vec{T} + \dot{C}(\vec{q})\right) \ . \tag{42}$$

In general, for any value of θ (assuming $d = 0$),

$$C(\vec{q})M^{-1}(\vec{q})\vec{T} = \frac{\rho}{2}\begin{bmatrix}\tau_1 - \tau_2 \\ 0 \\ (\tau_1 + \tau_2)/W\end{bmatrix} I_{yy}^{-1}; \qquad \dot{C}(\vec{q})\dot{\vec{q}} = \dot{\theta}\begin{bmatrix}\dot{x}\sin\theta - \dot{y}\cos\theta \\ \dot{x}\cos\theta + \dot{y}\sin\theta \\ 0\end{bmatrix} .$$

For $\theta = 0$,

$$C(\vec{q})M^{-1}(\vec{q})C(\vec{q})^T = \begin{bmatrix}\alpha & 0 & 0 \\ 0 & m_T^{-1} & 0 \\ 0 & 0 & \beta\end{bmatrix} \quad \text{where :}\ \alpha = m_T^{-1} + 2I_{yy}^{-1}\left(\frac{\rho}{2}\right)^2; \quad \beta = I_T^{-1} + 2I_{yy}^{-1}\left(\frac{\rho}{2}\right)^2; \tag{43}$$

Substituting these expression into Eq. (42) and evaluating at $\theta = 0$ yields:

$$\vec{\lambda} = -\frac{\rho}{2}I_{yy}^{-1}\begin{bmatrix}\alpha^{-1}(\tau_1 - \tau_2) \\ 0 \\ \beta^{-1}(\tau_1 + \tau_2)/W\end{bmatrix} + \dot{\theta}\begin{bmatrix}-\alpha^{-1}\dot{y} \\ m_T\dot{x} \\ 0\end{bmatrix} \tag{44}$$

Again, for the sake clarity in our understanding of the key design issues, let us assume that the vehicle is initially at rest, so that $\dot{x} = \dot{y} = 0$.

3.5.1 Analysis of Forward Acceleration

With the assumption that $\dot{x} = \dot{y} = 0$, the acceleration in the forward (or x) direction is governed by the equation:

$$m_T\ddot{x} = \lambda_1 = -\left(\frac{\rho}{2}\right)\frac{I_{yy}^{-1}}{m_T^{-1} + 2I_{yy}^{-1}\left(\frac{\rho}{2}\right)^2}(\tau_1 - \tau_2) \quad \Rightarrow \quad \ddot{x} = \lambda_1 = -\left(\frac{\rho}{2}\right)\frac{m_T^{-1}I_{yy}^{-1}}{m_T^{-1} + 2I_{yy}^{-1}\left(\frac{\rho}{2}\right)^2}(\tau_1 - \tau_2).$$

At first glance, this results suggests that to realize the largest forward acceleration, the wheel radius ρ should be made as large as possible, which is consistent with the conclusions from Section 2. However, recall that the rotational inertial of a cylindrical wheel about its axis of rotation is $I_{yy} = m_w\rho^2/2$. Substituting in this result, and rearranging, we arrive at the following simplified expression for the forward acceleration:

$$\ddot{x} \simeq -\left(\frac{1}{\rho}\right)\left(\frac{1}{m_w + m_T}\right)(\tau_1 - \tau_2). \tag{45}$$

If the wheel is not a solid cylinder, but instead you implement a wheel consisting largely of a rim, it's moment of inertial is still proportional to ρ^2, and an analogous results will hold.

Conclusions: Thus, this analysis suggests that to optimize forward acceleration

1. the wheel mass, m_w, and total mass, m_T, should be minimized. This is pretty obvious.

2. the maximum magnitudes of the wheel torques should be maximized. Maximum torque is optimized via the proper choice of the gear ratio. The higher the gear ratio (the greater the speed reduction), the higher the available torque. However, the high gear ratio also reduces the motor's (and hence the vehicle's) top end speed.

3. choose the smallest wheel radius that is practical. This result, which may be non-intuitive to some, directly contradicts the results of our kinematic analysis in Section 2. Hence, like your ME 71 transmission contest experience, it is necessary to determine the best trade-off between good acceleration and good top end speed.

3.5.2 Analysis of the Lateral Acceleration

The equation of motion in the y-direction are trivially simple:

$$m_T \dot{y} = \lambda_2 = 0 \ .$$

This result is not enlightening, but serves as a check on the validity of the derivations above– the vehicle should have no sideways acceleration since the no-slip constraints prevent direct sideways vehicle motion.

3.6 Analysis of the Rotational Acceleration

The equation of motion for the vehicle rotation is:

$$I_T \ddot{\theta} = -\left(\frac{\rho}{2}\right) \frac{I_{yy}^{-1} \beta^{-1} (\tau_1 + \tau_2)}{W} \tag{46}$$

Substituting in expressions and rearranging yields:

$$\ddot{\theta} = -\left(\frac{\rho}{2}\right) \frac{(\tau_1 + \tau_2)}{W I_T I_{yy} (I_T^{-1} + 2 I_{yy}^{-1} \left(\frac{\rho}{2W}\right)^2)} \tag{47}$$

$$= -\left(\frac{\rho}{2W}\right) \frac{\tau_1 + \tau_2}{I_{yy} + 2 I_T \left(\frac{\rho}{2W}\right)^2} \tag{48}$$

However, recall that the rotational inertial of the wheel is proportional to ρ^2: $I_{yy} \simeq \frac{m_w \rho^2}{2}$. Hence,

$$\ddot{\theta} \simeq -\left(\frac{W}{\rho}\right) \frac{\tau_1 + \tau_2}{m_w W^2 + I_T} = -\left(\frac{W}{\rho}\right) \frac{\tau_1 + \tau_2}{I_B + 2 I_{zz}^b + 3 m_w W^2} \tag{49}$$

Thus, to maximize angular acceleration ability (when the vehicle is operating at near zero velocity), one should:

1. minimize the wheel radius (just as in the case of forward acceleration)

2. minimize the wheel mass and total angular momentum of the main body (this is obvious!)

3. optimize the choice of the wheel base. Because $m_w < I_T$, the maximum angular acceleration will increase nearly linearly with increasing wheel base size for small W. However, at some point, the max acceleration will peak with increasing W, and then decrease thereafter. From the point of view of angular acceleration, the "optimal" wheel base can be calcuated by extremizing $\ddot{\theta}$ with respect to W:

$$\frac{\partial \ddot{\theta}}{\partial W} = -\frac{\tau_1 + \tau_2}{\rho} \left(\frac{I_B + 2I^b_{zz} - 3m_w W^2}{3m_2 W^2 + I_B + 2I^b_{zz}} \right) = 0 \tag{50}$$

Solving this equations for the optimal wheel base, W^*, yields:

$$W^* = \sqrt{\frac{I_B + 2I^b_{yy}}{3m_w}} \tag{51}$$

It is interesting to note that the optimal wheel base is independent of the wheel torque magnitudes or wheel radius. It only depends upon relative ratios of the total angular inertial and the wheel mass.