Weighted Line Fitting and Merging

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1 Problem Formulation

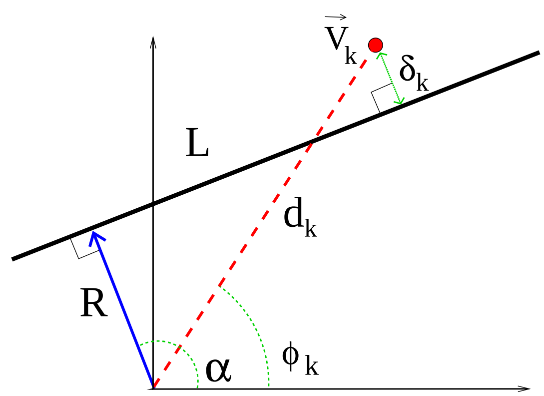
1.1 Line Representation

Line L is defined by the magnitude R and orientation α of the- vector from the origin normal to the line.

1.2 Point Measurements

A measured point $\vec{V}_k$ and the error δ_k between the point and line L(R,α) can be represented in the following ways:

1.2.1 Polar Form



with

$$\delta_k^p = d_k \cos(\alpha - \phi_k) - R \tag{1}$$

$$\vec{V}_k^p = \begin{bmatrix} d_k \cos \phi_k \\ d_k \sin \phi_k \end{bmatrix} \tag{2}$$

1.2.2 Cartesian Form

with

$$\begin{aligned} \delta_k^c &= d_k(\cos(\phi_k)\cos(\alpha) + \sin(\phi_k)\sin(\alpha)) - R \\ &= x_k \cos(\alpha) + y_k \sin(\alpha) - R \end{aligned} \tag{3}$$

$$\vec{V}_k^c = \begin{bmatrix} x_k \\ y_k \end{bmatrix} \tag{4}$$

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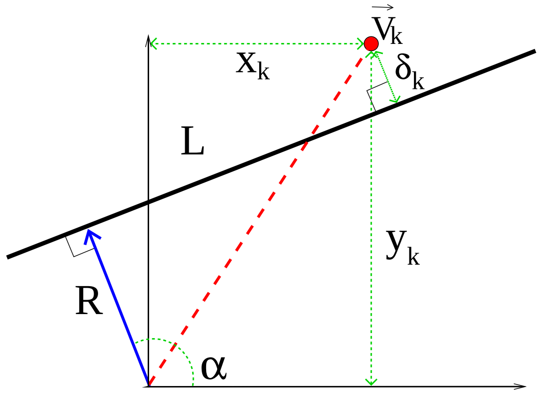


1.3 Covariance of the Virtual Measurements

(assuming that the line coordinates R,α are known).

1.3.1 Polar

Let

$$\epsilon_d = d_k - d_{0k} \tag{5}$$

where ϵ_d is the error of the measurement d_k. Similarly

$$\epsilon_\phi = \phi_k - \phi_{0k} \tag{6}$$

The virtual measurement with no error is defined as

$$\begin{aligned} \delta^p_{0k} &= d_{0k}\cos(\alpha - \phi_{0k}) - R \\ &= 0 \end{aligned} \tag{7}$$

For small ϵ_ϕ, ϵ_d we use the approximations:

$$\sin\epsilon_\phi \simeq \epsilon_\phi \tag{8}$$

$$\cos\epsilon_\phi \simeq 1 \tag{9}$$

$$\epsilon_\phi\epsilon_d \simeq 0 \tag{10}$$

We can then represent the virtual measurement δ^p_k as:

$$\begin{aligned} \delta^p_k &= d_k\cos(\alpha - \phi_k) - R \\ &= (d_{0k} + \epsilon_d)\cos(\alpha - \phi_{0k} - \epsilon_\phi) - R \\ &= (d_{0k} + \epsilon_d)(\cos(\alpha - \phi_{0k})\cos(\epsilon_\phi) + \sin(\alpha - \phi_{0k})\sin(\epsilon_\phi)) - R \\ &\simeq d_{0k}\cos(\alpha - \phi_{0k}) - R + \epsilon_d\cos(\alpha - \phi_{0k}) + d_{0k}\epsilon_\phi\sin(\alpha - \phi_{0k}) + \epsilon_d\epsilon_\phi\sin(\alpha - \phi_{0k}) \\ &\simeq 0 + \epsilon_d\cos(\alpha - \phi_{0k}) + d_{0k}\epsilon_\phi\sin(\alpha - \phi_{0k}) + 0 \\ &\simeq \epsilon_d\cos(\alpha - \phi_{0k}) + d_{0k}\epsilon_\phi\sin(\alpha - \phi_{0k}) \end{aligned} \tag{11}$$

The virtual measurement δ^p_k is assumed to be a zero-mean Gaussian process with

$$\begin{aligned} E\{\epsilon_\delta\} &= E\{\delta^p_k - \delta^p_{0k}\} = E\{\delta^p_k\} \\ &= E\{\epsilon_d\}\cos(\alpha - \phi_{0k}) + d_{0k}E\{\epsilon_\phi\}\sin(\alpha - \phi_{0k}) = 0 \\ P^p_{\delta_k} &= E\{\epsilon_\delta\epsilon_\delta^T\} = E\{\delta^p_k\delta^{pT}_k\} \\ &= E\{\epsilon_d\epsilon_d\}\cos^2(\alpha - \phi_{0k}) + E\{\epsilon_\phi\epsilon_\phi\}d^2_{0k}\sin^2(\alpha - \phi_{0k}) \\ &= \sigma^2_d\cos^2(\alpha - \phi_{0k}) + \sigma^2_\phi d^2_{0k}\sin^2(\alpha - \phi_{0k}) \end{aligned} \tag{12}$$

1.3.2 Cartesian

Let

$$\epsilon_x = x_k - x_{0k} \tag{13}$$

where ϵ_x is the error of the measurement x_k. Similarly

$$\epsilon_y = y_k - y_{0k} \tag{14}$$

The virtual measurement with no error is defined as

$$\begin{aligned} \delta_{0k}^c &= x_{0k}\cos(\alpha) + y_{0k}\sin(\alpha) - R \\ &= 0 \end{aligned} \tag{15}$$

For small ϵ_x, ϵ_y we use the approximation:

$$\epsilon_x \epsilon_y \simeq 0 \tag{16}$$

We can then represent the virtual measurement δ_k^c as:

$$\begin{aligned} \delta_k^c &= x_k\cos(\alpha) + y_k\sin(\alpha) - R \\ &= (x_{0k} + \epsilon_x)\cos(\alpha) + (y_{0k} + \epsilon_y)\sin(\alpha) - R \\ &= x_{0k}\cos(\alpha) + y_{0k}\sin(\alpha) - R + \epsilon_x\cos(\alpha) + \epsilon_y\sin(\alpha) \\ &= \epsilon_x\cos(\alpha) + \epsilon_y\sin(\alpha) \end{aligned} \tag{17}$$

The virtual measurement δ_k is assumed to be a zero-mean Gaussian process with

$$\begin{aligned} E\{\epsilon_\delta\} &= E\{\delta_k^c - \delta_{0k}^c\} = E\{\delta_k^c\} \\ &= E\{\epsilon_x\}\cos(\alpha) + E\{\epsilon_y\}\sin(\alpha) = 0 \\ P_{\delta_k}^c &= E\{\epsilon_\delta \epsilon_\delta^T\} = E\{\delta_k^c \delta_k^{cT}\} \\ &= E\{\epsilon_x\epsilon_x\}\cos^2(\alpha) + E\{\epsilon_y\epsilon_y\}\sin^2(\alpha) + 2E\{\epsilon_x\epsilon_y\}\cos(\alpha)\sin(\alpha) \\ &= \sigma_x^2\cos^2(\alpha) + \sigma_y^2\sin^2(\alpha) + 2\sigma_{xy}\cos(\alpha)\sin(\alpha) \end{aligned} \tag{18}$$

1.4 Determination of $\chi^2(L)$ cost function

In order to estimate the parameters R, α we have to minimize the quantity:

1.4.1 Polar

$$\begin{aligned} \chi^2(L) &= \sum_{k=1}^{N} \frac{(\delta_k^p)^2}{P_{\delta_k}^p} \\ &= \sum_{k=1}^{N} \frac{(d_k\cos(\alpha - \phi_k) - R)^2}{\sigma_d^2\cos^2(\alpha - \phi_{0k}) + \sigma_\phi^2 d_{0k}^2\sin^2(\alpha - \phi_{0k})} \end{aligned} \tag{19}$$

where

$$L = \begin{bmatrix} R \\ \alpha \end{bmatrix} \tag{20}$$

is the unknown parameter vector.

1.4.2 Cartesian

$$\begin{aligned}
\chi^2(L) &= \sum_{k=1}^{N} \frac{(\delta_k^c)^2}{P_{\delta_k}^c} \\
&= \sum_{k=1}^{N} \frac{(x_k \cos(\alpha) + y_k \sin(\alpha) - R)^2}{\sigma_x^2 \cos^2(\alpha) + \sigma_y^2 \sin^2(\alpha) + 2\sigma_{xy} \cos(\alpha) \sin(\alpha)}
\end{aligned} \tag{21}$$

where

$$L = \begin{bmatrix} R \\ \alpha \end{bmatrix} \tag{22}$$

is the unknown parameter vector.

If an initial estimate of L, $\widehat{L} = \begin{bmatrix} \widehat{R} & \widehat{\alpha} \end{bmatrix}^T$ is available, we can use it to minimize $\chi^2(L)$ in two iterative steps: (i) for the displacement R given $\widehat{\alpha}$ and (ii) for the orientation displacement α given $\widehat{R}$, $\widehat{\alpha}$.

2 Distance to Line Estimation

2.1 Minimization of χ^2 with respect to R

Given an estimate of the heading to the line $\hat{\alpha}$, Eqs. (19) and (21) can be written in terms of the unknown R.

2.1.1 Polar

$$\chi^2(R) = \sum_{k=1}^{N} \frac{(d_k \cos(\hat{\alpha} - \phi_k) - R)^2}{\sigma_d^2 \cos^2(\hat{\alpha} - \phi_{0k}) + \sigma_\phi^2 d_{0k}^2 \sin^2(\hat{\alpha} - \phi_{0k})} = \sum_{k=1}^{N} \frac{(\delta_k^p)^2}{P_{\delta_k}^p} \tag{23}$$

In order to minimize Eq. (23) we have to set

$$\begin{aligned}
\frac{\partial \chi^2(R)}{\partial (R)} &= 0 \Leftrightarrow \\
\sum_{k=1}^{N} \frac{(-2)(d_k \cos(\hat{\alpha} - \phi_k) - R)}{P_{\delta_k}^p} &= 0 \Leftrightarrow \\
\sum_{k=1}^{N} \frac{d_k \cos(\hat{\alpha} - \phi_k)}{P_{\delta_k}^p} - \sum_{k=1}^{N} \frac{R}{P_{\delta_k}^p} &= 0 \Leftrightarrow \\
\sum_{k=1}^{N} \frac{d_k \cos(\hat{\alpha} - \phi_k)}{P_{\delta_k}^p} &= R \left(\sum_{k=1}^{N} \frac{1}{P_{\delta_k}^p} \right)
\end{aligned}$$

or

$$R = \frac{\sum_{k=1}^{N} \frac{d_k \cos(\hat{\alpha} - \phi_k)}{P_{\delta_k}^p}}{\sum_{k=1}^{N} \frac{1}{P_{\delta_k}^p}} \tag{24}$$

where, from Eq. (12)

$$P_{\delta_k}^p = \sigma_d^2 \cos^2(\hat{\alpha} - \phi_{0k}) + \sigma_\phi^2 d_{0k}^2 \sin^2(\hat{\alpha} - \phi_{0k}) \tag{25}$$

2.1.2 Cartesian

$$\chi^2(R) = \sum_{k=1}^{N} \frac{(x_k \cos(\hat{\alpha}) + y_k \sin(\hat{\alpha}) - R)^2}{\sigma_x^2 \cos^2(\hat{\alpha}) + \sigma_y^2 \sin^2(\hat{\alpha}) + 2\sigma_{xy} \cos(\hat{\alpha}) \sin(\hat{\alpha})} = \sum_{k=1}^{N} \frac{(\delta_k^c)^2}{P_{\delta_k}^c} \tag{26}$$

In order to minimize Eq. (26) we have to set

$$\begin{aligned}
\frac{\partial \chi^2(R)}{\partial (R)} &= 0 \Leftrightarrow \\
\sum_{k=1}^{N} \frac{(-2)(x_k \cos(\hat{\alpha}) + y_k \sin(\hat{\alpha}) - R)}{P_{\delta_k}^c} &= 0 \Leftrightarrow \\
\sum_{k=1}^{N} \frac{x_k \cos(\hat{\alpha}) + y_k \sin(\hat{\alpha})}{P_{\delta_k}^c} - \sum_{k=1}^{N} \frac{R}{P_{\delta_k}^c} &= 0 \Leftrightarrow \\
\sum_{k=1}^{N} \frac{x_k \cos(\hat{\alpha}) + y_k \sin(\hat{\alpha})}{P_{\delta_k}^c} &= R \left(\sum_{k=1}^{N} \frac{1}{P_{\delta_k}^c} \right)
\end{aligned}$$

or

$$R = \frac{\sum_{k=1}^{N} \frac{x_k \cos(\hat{\alpha}) + y_k \sin(\hat{\alpha})}{P_{\delta_k}^c}}{\sum_{k=1}^{N} \frac{1}{P_{\delta_k}^c}} \tag{27}$$

where, from Eq. (18)

$$P_{\delta_k}^c = \sigma_x^2 \cos^2(\hat{\alpha}) + \sigma_y^2 \sin^2(\hat{\alpha}) + 2\sigma_{xy} \cos(\hat{\alpha}) \sin(\hat{\alpha}) \tag{28}$$

3 Heading to Line Estimation

3.1 Minimization of χ^2 with respect to α

Given an estimate of the distance to the line $\hat{R}$, Eqs. (19) and (21) can be written in terms of the unknown heading to line parameter α.

3.2 Polar

$$\chi^2(\alpha) = \sum_{k=1}^{N} \frac{(d_k \cos(\alpha - \phi_k) - \hat{R})^2}{\sigma_d^2 \cos^2(\alpha - \phi_{0k}) + \sigma_\phi^2 d_{0k}^2 \sin^2(\alpha - \phi_{0k})} = \sum_{k=1}^{N} \frac{(\delta_k^p)^2}{P_{\delta_k}^p} \tag{29}$$

In order to minimize Eq. (29) we have to set

$$\frac{\partial \chi^2(\alpha)}{\partial \alpha} = 0 \tag{30}$$

Instead of directly computing the gradient of $\chi^2(\alpha)$ with respect to α, we will calculate it as follows:

$$\frac{\partial(\chi^2(\alpha))}{\partial \alpha} = \frac{\partial(\chi^2(\widehat{\alpha} + \delta\alpha))}{\partial(\delta\alpha)} \frac{\partial(\delta\alpha)}{\partial \alpha} = \frac{\partial(\chi^2(\delta\alpha))}{\partial(\delta\alpha)} \frac{1}{\frac{\partial \alpha}{\partial(\delta\alpha)}} = \frac{\partial(\chi^2(\delta\alpha))}{\partial(\delta\alpha)} \tag{31}$$

where we used the relation:

$$\alpha = \widehat{\alpha} + \delta\alpha \Rightarrow \frac{\partial \alpha}{\partial(\delta\alpha)} = 1 \tag{32}$$

so

$$\chi^2(\alpha) = \chi^2(\widehat{\alpha} + \delta\alpha) = \sum_{k=0}^{N} G_k(\delta\alpha) \tag{33}$$

with

$$G_k(\delta\alpha) = \frac{(d_k \cos(\widehat{\alpha} + \delta\alpha - \phi_k) - \hat{R})^2}{\sigma_d^2 \cos^2(\widehat{\alpha} + \delta\alpha - \phi_k) + \sigma_\phi^2 d_k^2 \sin^2(\widehat{\alpha} + \delta\alpha - \phi_k)} \tag{34}$$

Applying Taylor series approximation to $G_k(\delta\alpha)$ we have:

$$G_k(\delta\alpha) = G_k(0) + \frac{1}{1!} G_k^{'}(0)\delta\alpha + \frac{1}{2!} G_k^{''}(0)\delta\alpha^2 + \frac{1}{3!} G_k^{'''}(0)\delta\alpha^3 + ... \tag{35}$$

let

$$\begin{aligned}
c_k &= \cos(\widehat{\alpha} + \delta\alpha - \phi_k) \\
s_k &= \sin(\widehat{\alpha} + \delta\alpha - \phi_k) \\
a_k(\delta\alpha) &= (d_k c_k - \hat{R})^2 \\
a_k^{'}(\delta\alpha) &= \frac{\partial a_k(\delta\alpha)}{\partial \delta\alpha} = -2 d_k s_k (d_k c_k - \hat{R}) \\
a_k^{''}(\delta\alpha) &= \frac{\partial^2 a_k(\delta\alpha)}{(\partial \delta\alpha)^2} = 2 d_k^2 s_k^2 - 2 d_k c_k (d_k c_k - \hat{R}) \\
b_k(\delta\alpha) &= \sigma_d^2 c_k^2 + \sigma_\phi^2 d_k^2 s_k^2 \\
b_k^{'}(\delta\alpha) &= \frac{\partial b_k(\delta\alpha)}{\partial \delta\alpha} = 2(d_k^2 \sigma_\phi^2 - \sigma_d^2) c_k s_k \\
b_k^{''}(\delta\alpha) &= \frac{\partial^2 b_k(\delta\alpha)}{(\partial \delta\alpha)^2} = 2(d_k^2 \sigma_\phi^2 - \sigma_d^2)(c_k^2 - s_k^2)
\end{aligned} \tag{36}$$

so

$$G_k(0) = \frac{a_k(0)}{b_k(0)} \tag{37}$$

and

$$G'_k(0) = \frac{b_k(0)a'_k(0) - a_k(0)b'_k(0)}{(b_k(0))^2} \tag{38}$$

and

$$\begin{aligned} G''_k(0) &= \frac{-2a'_k(0)b'_k(0)(b_k(0))^2 + a''_k(0)(b_k(0))^3 + 2a_k(0)b_k(0)(b'_k(0))^2 - a_k(0)(b_k(0))^2 b''_k(0)}{(b_k(0))^4} \\ &= \frac{\left(a''_k(0)b_k(0) - a_k(0)b''_k(0)\right) b_k(0) - 2\left(a'_k(0)b_k(0) - a_k(0)b'_k(0)\right) b'_k(0)}{(b_k(0))^3} \end{aligned} \tag{39}$$

so

$$\chi^2(\delta\alpha) = \sum_{k=1}^{N} \{G_k(0) + \frac{1}{1!}G'_k(0)\delta\alpha + \frac{1}{2!}G''_k(0)\delta\alpha^2 + \frac{1}{3!}G'''_k(0)\delta\alpha^3 + ...\} \tag{40}$$

Note: There is *no* approximation made up to this point. The previous equation is the complete analytical expression of the cost function. It is expressed as an infinite series of polynomial terms of the orientation estimation error $\delta\alpha$. In order to minimize this function we have to approximate it after considering a limited number of terms.

3.2.1 Second Order Approximation

$$\frac{\partial(\chi^2(\delta\alpha))}{\partial(\delta\alpha)} \simeq \sum_{k=1}^{N} \{\left[G'_k(0)\right] + \frac{2}{2!}\left[G''_k(0)\right]\delta\alpha\} \tag{41}$$

Finally, by substituting the previous expression with Eq. (38) and Eq. (39) in Eq. (30) and solving for $\delta\alpha$ we have:

$$\delta\alpha = -\frac{\sum_{k=1}^{N} G'_k(0)}{\sum_{k=1}^{N} G''_k(0)} \tag{42}$$

4 Covariance Estimation

Let the covariance for line L be defined as:

$$P_L = \begin{bmatrix} P_{RR} & P_{R\alpha} \\ P_{\alpha R} & P_{\alpha\alpha} \end{bmatrix} \tag{43}$$

with distance to line covariance P_{RR}, heading to line covariance $P_{\alpha\alpha}$ and cross-correlation covariance terms $P_{R\alpha} = P_{\alpha R}$ derived in the following sections.

4.1 Distance to Line Estimate Covariance

To estimate R define the following:

$$R = g_R(Y)$$

where

$$Y = \begin{bmatrix} Y_1 \\ . \\ . \\ Y_N \end{bmatrix}$$

and

$$\begin{aligned} P_{RR} &= E\{\epsilon_R \epsilon_R^T\} \\ &= E\{\sum_{k=1}^{N} \left((\nabla_{Y_k}^T g_R \; \epsilon_{Y_k}) \; (\nabla_{Y_k}^T g_R \; \epsilon_{Y_k})^T \right)\} \\ &= \sum_{k=1}^{N} \left((\nabla_{Y_k}^T g_R) \;\; E\{\epsilon_{Y_k} \epsilon_{Y_k}^T\} \; (\nabla_{Y_k}^T g_R)^T \right) \\ &= \sum_{k=1}^{N} \left((\nabla_{Y_k}^T g_R) \, P_{Y_k Y_k} \, (\nabla_{Y_k} g_R) \right) \end{aligned} \tag{44}$$

4.1.1 Polar

$$Y_k = \begin{bmatrix} d_k \\ \phi_k \end{bmatrix} \tag{45}$$

with

$$P_{Y_k Y_k} = E\{\epsilon_{Y_k} \epsilon_{Y_k}^T\} = E\{\begin{bmatrix} \epsilon_{d_k} \\ \epsilon_{\phi_k} \end{bmatrix} \begin{bmatrix} \epsilon_{d_k} & \epsilon_{\phi_k} \end{bmatrix}\} = \begin{bmatrix} \sigma_d^2 & 0 \\ 0 & \sigma_\phi^2 \end{bmatrix} \tag{46}$$

From Eq. (24):

$$g_R = R = \frac{\sum_{k=1}^{N} \frac{d_k \cos(\hat{\alpha} - \phi_k)}{P_{\delta_k}^p}}{\sum_{k=1}^{N} \frac{1}{P_{\delta_k}^p}} \tag{47}$$

and therefore

$$\nabla_{d_k} g_R = \frac{\frac{\cos(\hat{\alpha} - \phi_k)}{P_{\delta_k}^p}}{\sum_{j=1}^{N} \frac{1}{P_{\delta_j}^p}} \tag{48}$$

$$\nabla_{\phi_k} g_R = \frac{\frac{d_k \sin(\hat{\alpha} - \phi_k)}{P_{\delta_k}^p}}{\sum_{k=j}^{N} \frac{1}{P_{\delta_j}^p}} \tag{49}$$

so

$$
\begin{aligned}
P_{RR} &= \sum_{k=1}^{N} \left(\nabla_{Y_k}^T g_R\right) P_{Y_k Y_k} \left(\nabla_{Y_k} g_R\right) \\
&= \sum_{k=1}^{N} \left[\begin{array}{c|c} \nabla_{d_k} g_R & \nabla_{\phi_k} g_R \end{array} \right] \left[\begin{array}{cc} \sigma_d^2 & 0 \\ 0 & \sigma_\phi^2 \end{array} \right] \left[\begin{array}{c} \nabla_{d_k} g_R \\ - \\ \nabla_{\phi_k} g_R \end{array} \right] \\
&= \sum_{k=1}^{N} (\nabla_{d_k} g_R)^2 \sigma_d^2 + \sum_{k=1}^{N} (\nabla_{\phi_k} g_R)^2 \sigma_\phi^2 \\
&= \sum_{k=1}^{N} \left(\frac{\frac{\cos(\hat{\alpha}-\phi_k)}{P_{\delta_k}^p}}{\sum_{k=1}^{N} \frac{1}{P_{\delta_k}^p}} \right)^2 \sigma_d^2 + \sum_{k=1}^{N} \left(\frac{\frac{d_k \sin(\hat{\alpha}-\phi_k)}{P_{\delta_k}^p}}{\sum_{k=1}^{N} \frac{1}{P_{\delta_k}^p}} \right)^2 \sigma_\phi^2 \\
&= \sum_{k=1}^{N} \left(\frac{\frac{\cos^2(\hat{\alpha}-\phi_k)}{(P_{\delta_k}^p)^2}}{\left(\sum_{k=1}^{N} \frac{1}{P_{\delta_k}^p}\right)^2} \sigma_d^2 + \frac{\frac{d_k^2 \sin^2(\hat{\alpha}-\phi_k)}{(P_{\delta_k}^p)^2}}{\left(\sum_{k=1}^{N} \frac{1}{P_{\delta_k}^p}\right)^2} \sigma_\phi^2 \right) \\
&= \frac{\sum_{k=1}^{N} \frac{\sigma_d^2 \cos^2(\hat{\alpha}-\phi_k) + \sigma_\phi^2 d_k^2 \sin^2(\hat{\alpha}-\phi_k)}{(P_{\delta_k}^p)^2}}{\left(\sum_{k=1}^{N} \frac{1}{P_{\delta_k}^p}\right)^2} \\
&= \frac{\sum_{k=1}^{N} \frac{1}{P_{\delta_k}^p}}{\left(\sum_{k=1}^{N} \frac{1}{P_{\delta_k}^p}\right)^2} \\
&= \frac{1}{\left(\sum_{k=1}^{N} \frac{1}{P_{\delta_k}^p}\right)}
\end{aligned} \tag{50}
$$

with $P_{\delta_k}^p$ defined in Eqn. (12)

4.1.2 Cartesian

$$
Y_k = \left[\begin{array}{c} x_k \\ y_k \end{array} \right] \tag{51}
$$

with

$$
P_{Y_k Y_k} = E\{\epsilon_{Y_k} \epsilon_{Y_k}^T\} = E\{\left[\begin{array}{c} \epsilon_{x_k} \\ \epsilon_{y_k} \end{array} \right] \left[\begin{array}{cc} \epsilon_{x_k} & \epsilon_{y_k} \end{array} \right]\} = \left[\begin{array}{cc} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{array} \right] \tag{52}
$$

From Eq. (27):

$$
g_R = R = \frac{\sum_{k=1}^{N} \frac{x_k \cos(\hat{\alpha}) + y_k \sin(\hat{\alpha})}{P_{\delta_k}^c}}{\sum_{k=1}^{N} \frac{1}{P_{\delta_k}^c}} \tag{53}
$$

and therefore

$$
\nabla_{x_k} g_R = \frac{\frac{\cos(\hat{\alpha})}{P_{\delta_k}^c}}{\sum_{j=1}^{N} \frac{1}{P_{\delta_j}^c}} \tag{54}
$$

$$
\nabla_{y_k} g_R = \frac{\frac{\sin(\hat{\alpha})}{P_{\delta_k}^c}}{\sum_{k=j}^{N} \frac{1}{P_{\delta_j}^c}} \tag{55}
$$

so

$$
\begin{aligned}
P_{RR} &= \sum_{k=1}^{N} \left(\nabla_{Y_k}^T g_R\right) P_{Y_k Y_k} \left(\nabla_{Y_k} g_R\right) \\
&= \sum_{k=1}^{N} \left[\begin{array}{c|c} \nabla_{x_k} g_R & \nabla_{y_k} g_R \end{array}\right] \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \begin{bmatrix} \nabla_{x_k} g_R \\ - \\ \nabla_{y_k} g_R \end{bmatrix} \\
&= \sum_{k=1}^{N} (\nabla_{x_k} g_R)^2 \sigma_x^2 + \sum_{k=1}^{N} (\nabla_{y_k} g_R)^2 \sigma_y^2 \\
&= \sum_{k=1}^{N} \left(\frac{\frac{\cos(\hat{\alpha})}{P^c_{\delta_k}}}{\sum_{k=1}^{N} \frac{1}{P^c_{\delta_k}}} \right)^2 \sigma_d^2 + \sum_{k=1}^{N} \left(\frac{\frac{d_k \sin(\hat{\alpha})}{P^c_{\delta_k}}}{\sum_{k=1}^{N} \frac{1}{P^c_{\delta_k}}} \right)^2 \sigma_\phi^2 \\
&= \sum_{k=1}^{N} \left(\frac{\frac{\cos^2(\hat{\alpha})}{(P^c_{\delta_k})^2}}{\left(\sum_{k=1}^{N} \frac{1}{P^c_{\delta_k}}\right)^2} \sigma_x^2 + \frac{\frac{\sin^2(\hat{\alpha})}{(P^c_{\delta_k})^2}}{\left(\sum_{k=1}^{N} \frac{1}{P^c_{\delta_k}}\right)^2} \sigma_y^2 \right) \\
&= \frac{\sum_{k=1}^{N} \frac{\sigma_x^2 \cos^2(\hat{\alpha}) + \sigma_y^2 \sin^2(\hat{\alpha})}{(P^c_{\delta_k})^2}}{\left(\sum_{k=1}^{N} \frac{1}{P^c_{\delta_k}}\right)^2} \\
&= \frac{\sum_{k=1}^{N} \frac{1}{P^c_{\delta_k}}}{\left(\sum_{k=1}^{N} \frac{1}{P^c_{\delta_k}}\right)^2} \\
&= \frac{1}{\left(\sum_{k=1}^{N} \frac{1}{P^c_{\delta_k}}\right)} \qquad (56)
\end{aligned}
$$

with $P^c_{\delta_k}$ defined in Eqn. (18)

4.2 Heading to Line Estimate Covariance

$$
Y_k = \begin{bmatrix} d_k \\ \phi_k \end{bmatrix} \qquad (57)
$$

$$
\begin{aligned}
P_{\alpha\alpha} &= E\{\epsilon_\alpha \epsilon_\alpha^T\} \\
&= E\{\left(\nabla_Y^T g_\alpha \ \epsilon_Y\right) \left(\nabla_Y^T g_\alpha \ \epsilon_Y\right)^T\} \\
&= \left(\nabla_Y^T g_\alpha\right) \ E\{\epsilon_Y \epsilon_Y^T\} \left(\nabla_Y^T g_\alpha\right)^T \\
&= \left(\nabla_Y^T g_\alpha\right) P_{YY} \left(\nabla_Y g_\alpha\right) \qquad (58)
\end{aligned}
$$

From Eqs. (38), (39), (42)

$$
\begin{aligned}
g_\alpha &= \alpha = \widehat{\alpha} + \delta\alpha \\
&= \widehat{\alpha} - \frac{\sum_{k=1}^{N} G_k'(0)}{\sum_{k=1}^{N} G_k''(0)} \\
&= \widehat{\alpha} - \frac{\sum_{k=1}^{N} \frac{b_k a_k' - a_k b_k'}{(b_k')^2}}{\sum_{k=1}^{N} \frac{\left(a_k'' b_k - a_k b_k''\right) b_k - 2\left(a_k' b_k - a_k b_k'\right) b_k'}{(b_k)^3}} \qquad (59)
\end{aligned}
$$

where

$$a_k = a_k(0) = (d_k \cos(\widehat{\alpha} - \phi_k) - \hat{R})^2 \tag{60}$$
$$b_k = b_k(0) = \sigma_d^2 \cos^2(\widehat{\alpha} - \phi_k) + \sigma_\phi^2 d_k^2 \sin^2(\widehat{\alpha} - \phi_k) \tag{61}$$

and $\widehat{\alpha}$ is a constant (the current estimate of orientation computed in the last step of the ML algorithm) so

$$\begin{aligned}
\nabla_{d_k} g_\alpha &= -\left(\frac{\left(\nabla_{d_k}\left(\sum_{j=1}^N G_j'(0)\right)\right)\left(\sum_{j=1}^N G_j''(0)\right) - \left(\sum_{j=1}^N G_j'(0)\right)\left(\nabla_{d_k}\left(\sum_{j=1}^N G_j''(0)\right)\right)}{\left(\sum_{j=1}^N G_j''(0)\right)^2} \right) \\
&= \frac{-\nabla_{d_k}(G_k'(0))\left(\sum_{j=1}^N G_j''(0)\right) + \left(\sum_{j=1}^N G_j'(0)\right)\nabla_{d_k}(G_k''(0))}{\left(\sum_{j=1}^N G_j''(0)\right)^2} \\
&= -\frac{1}{G_T''}\nabla_{d_k}(G_k'(0)) + \frac{G_T'}{(G_T'')^2}\nabla_{d_k}(G_k''(0))
\end{aligned} \tag{62}$$

where

$$G_T' = \sum_{j=1}^N G_j'(0)\ , \quad G_T'' = \sum_{j=1}^N G_j''(0)$$

Similarly

$$\nabla_{\phi_k} g_\alpha = -\frac{1}{G_T''}\nabla_{\phi_k}(G_k'(0)) + \frac{G_T'}{(G_T'')^2}\nabla_{\phi_k}(G_k''(0)) \tag{63}$$

and From Eq. (58), substituting from Eqs. (46), (62), (63):

$$\begin{aligned}
P_{\alpha\alpha} &= \sum_{k=1}^N \left(\nabla_{Y_k}^T g_\alpha\right) P_{Y_k Y_k} \left(\nabla_{Y_k} g_\alpha\right) \\
&= \sum_{k=1}^N \left[\ \nabla_{d_k} g_\alpha \quad | \quad \nabla_{\phi_k} g_\alpha\ \right] \begin{bmatrix} \sigma_d^2 & 0 \\ 0 & \sigma_\phi^2 \end{bmatrix} \begin{bmatrix} \nabla_{d_k} g_\alpha \\ - \\ \nabla_{\phi_k} g_\alpha \end{bmatrix} \\
&= \sum_{k=1}^N (\nabla_{d_k} g_\alpha)^2 \sigma_d^2 + \sum_{k=1}^N (\nabla_{\phi_k} g_\alpha)^2 \sigma_\phi^2 \\
&= \sum_{k=1}^N \left(-\frac{1}{G_T''}\nabla_{d_k}(G_k'(0)) + \frac{G_T'}{(G_T'')^2}\nabla_{d_k}(G_k''(0)) \right)^2 \sigma_d^2 \\
&\quad + \sum_{k=1}^N \left(-\frac{1}{G_T''}\nabla_{\phi_k}(G_k'(0)) + \frac{G_T'}{(G_T'')^2}\nabla_{\phi_k}(G_k''(0)) \right)^2 \sigma_\phi^2
\end{aligned} \tag{64}$$

4.2.1 Complete $G_k'(0)$ $G_k''(0)$

Omitting the index k we start from Eqs. (38), (39)

$$G_k'(0) = \frac{ba' - ab'}{b^2}$$

and

$$G_k''(0) = \frac{\left(a''b - ab''\right)b - 2\left(a'b - ab'\right)b'}{b^3}$$

with

$$\begin{aligned}
c &= \cos(\widehat{\alpha} - \phi) \\
s &= \sin(\widehat{\alpha} - \phi) \\
a &= (dc - \hat{R})^2 \\
a^{'} &= \frac{\partial a(\delta\alpha)}{\partial\delta\alpha} = -2ds(dc - \hat{R}) \\
a^{''} &= \frac{\partial^2 a(\delta\alpha)}{(\partial\delta\alpha)^2} = 2d^2s^2 - 2dc(dc - \hat{R}) \\
b &= \sigma_d^2 c^2 + \sigma_\phi^2 d^2 s^2 \\
b^{'} &= \frac{\partial b(\delta\alpha)}{\partial\delta\alpha} = 2(d^2\sigma_\phi^2 - \sigma_d^2)cs \\
b^{''} &= \frac{\partial^2 b(\delta\alpha)}{(\partial\delta\alpha)^2} = 2(d^2\sigma_\phi^2 - \sigma_d^2)(c^2 - s^2)
\end{aligned} \tag{65}$$

we can calculate

$$\begin{aligned}
\nabla_d(G^{'}(0)) &= \frac{\partial(G^{'}(0))}{\partial d} \\
&= \frac{\partial\left(\frac{ba^{'} - ab^{'}}{b^2}\right)}{\partial d} \\
&= \frac{(a^{'}b_d + a^{'}_d b - b^{'}a_d - b^{'}_d a)b - (a^{'}b - b^{'}a)2b_d}{b^3}
\end{aligned} \tag{66}$$

$$\begin{aligned}
\nabla_\phi(G^{'}(0)) &= \frac{\partial(G^{'}(0))}{\partial\phi} \\
&= \frac{\partial\left(\frac{ba^{'} - ab^{'}}{b^2}\right)}{\partial\phi} \\
&= \frac{(a^{'}b_\phi + a^{'}_\phi b - b^{'}a_\phi - b^{'}_\phi a)b - (a^{'}b - b^{'}a)2b_\phi}{b^3}
\end{aligned} \tag{67}$$

$$\begin{aligned}
\nabla_d(G^{''}(0)) &= \frac{\partial(G^{''}(0))}{\partial d} \\
&= \frac{\partial\left(\frac{\left(a^{''}b - ab^{''}\right)b - 2\left(a^{'}b - a_k b^{'}_k\right)b^{'}}{b^3}\right)}{\partial d}
\end{aligned} \tag{68}$$

$$\begin{aligned}
\nabla_d(G^{''}(0)) &= \frac{\partial(G^{''}(0))}{\partial\phi} \\
&= \frac{\partial\left(\frac{\left(a^{''}b - ab^{''}\right)b - 2\left(a^{'}b - a_k b^{'}_k\right)b^{'}}{b^3}\right)}{\partial\phi}
\end{aligned} \tag{69}$$

4.2.2 Approximate $G^{'}_k(0)$ $G^{''}_k(0)$

Assume small errors such that $|\delta| << |r|$, i.e. the distance from a point to the line is small compared to the distance from that point to the origin, where

$$|\delta| = |r\cos(\alpha - \phi) - R|$$

$$|r| = \|\vec{V}\| = \| \begin{bmatrix} d_k \cos\phi_k \\ d_k \sin\phi_k \end{bmatrix} \|$$

given

$$\begin{aligned}
a &= (dc - \hat{R})^2 = \delta^2 \sim O(\delta^2) && (70)\\
a^{'} &= -2ds(dc - \hat{R}) = -2ds(\delta) \sim O(d\delta) && (71)\\
a_d &= 2c(dc - \hat{R}) = 2c\delta \sim O(\delta) && (72)\\
a_d^{'} &= -2s(dc - \hat{R}) - 2dsc = -2s(\delta) - 2dsc \sim O(d) && (73)\\
b &= \sigma_d^2 c^2 + \sigma_\phi^2 d^2 s^2 \sim O(d^2) && (74)\\
b^{'} &= 2(d^2\sigma_\phi^2 - \sigma_d^2)cs \sim O(d^2) && (75)\\
b_d &= 2\sigma_\phi d s^2 \sim O(d) && (76)\\
b_d^{'} &= 4d\sigma_\phi^2 cs \sim O(d) && (77)
\end{aligned}$$

it can be approximated that

$$\begin{aligned}
a_d^{'} &>> a\\
a_d^{'} &>> a^{'}\\
a_d^{'} &>> a_d
\end{aligned}$$

so

$$\begin{aligned}
\nabla_d(G^{'}(0)) &= \frac{(a^{'}b_d + a_d^{'}b - b^{'}a_d - b_d^{'}a)b - (a^{'}b - b^{'}a)2b_d}{b^3}\\
&\simeq \frac{a_d^{'}b^2}{b^3}\\
&\simeq \frac{a_d^{'}}{b} && (78)
\end{aligned}$$

with

$$\begin{aligned}
a_d^{'} &= \frac{\partial\left(-2d\sin(\alpha-\phi)(d\cos(\alpha-\phi) - R)\right)}{\partial d}\\
&= -2d\cos(\alpha-\phi)\sin(\alpha-\phi) - 2(d\cos(\alpha-\phi) - R)\sin(\alpha-\phi)\\
&= -2d\cos(\alpha-\phi)\sin(\alpha-\phi) - 2\delta\sin(\alpha-\phi)\\
&\simeq -2d\cos(\alpha-\phi)\sin(\alpha-\phi) && (79)
\end{aligned}$$

Similarly

$$\begin{aligned}
a_\phi^{'} &>> a\\
a_\phi^{'} &>> a^{'}\\
a_\phi^{'} &>> a_\phi
\end{aligned}$$

so

$$\nabla_\phi(G^{'}(0)) \simeq \frac{a_\phi^{'}}{b} \qquad (80)$$

with

$$\begin{aligned}
a_\phi^{'} &= \frac{\partial\left(-2d\sin(\alpha-\phi)(d\cos(\alpha-\phi) - R)\right)}{\partial\phi}\\
&= -2d^2\sin^2(\alpha-\phi) + 2(d\cos(\alpha-\phi) - R)d\cos(\alpha-\phi)\\
&= -2d^2\sin^2(\alpha-\phi) + 2\delta d\sin(\alpha-\phi)\\
&\simeq -2d^2\sin^2(\alpha-\phi) && (81)
\end{aligned}$$

From Eqs. (38), (39) :

$$G_k^{'}(0) = \frac{ba^{'} - ab^{'}}{b^2}$$

and

$$G_k^{''}(0) = \frac{\left(a^{''}b - ab^{''}\right)b - 2\left(a^{'}b - ab^{'}\right)b^{'}}{b^3}$$

Consider Eqs. (70) - (77) and

$$a^{''} = -2dc(dc - \hat{R}) + 2d^2s^2 = -2dc\delta + 2d^2s^2 \sim O(d^2) \tag{82}$$

$$b^{''} = 2(d^2\sigma_\phi^2 - \sigma_d^2)(c^2 - s^2) \sim O(d^2) \tag{83}$$

we can then show that

$$G_k^{'}(0) \sim \frac{O(d^3\delta) - O(d^2\delta^2)}{O(d^4)} \sim O(\delta/d) \tag{84}$$

$$G_k^{''}(0) \sim \frac{\left(O(d^4) - O(d^2\delta^2)\right)O(d^2) - 2\left(O(d^3\delta) - O(d^2\delta^2)\right)O(d^2)}{O(d^6)} \tag{85}$$

$$\sim O(d/d) \tag{86}$$

so

$$G_k^{''}(0) >> G_k^{'}(0) \Rightarrow G_T^{''} >> G_T^{'}$$
$$\Rightarrow \frac{G_T^{'}}{(G_T^{''})^2} \simeq 0 \tag{87}$$

so from Eqs. (78), (80), (81) and (87) we can approximate Eq. (64) as

$$\begin{aligned}
P_{\alpha\alpha} &\simeq \sum_{k=1}^{N}\left(-\frac{1}{G_T^{''}}\nabla_{d_k}(G_k^{'}(0))\right)^2\sigma_d^2 \\
&\quad + \sum_{k=1}^{N}\left(-\frac{1}{G_T^{''}}\nabla_{\phi_k}(G_k^{'}(0))\right)^2\sigma_\phi^2 \\
&= \frac{1}{(G_T^{''})^2}\sum_{k=1}^{N}\left(-\frac{a_{di}^{'}}{b_k}\right)^2\sigma_d^2 + \frac{1}{(G_T^{''})^2}\sum_{k=1}^{N}\left(-\frac{a_{\phi i}^{'}}{b_k}\right)^2\sigma_\phi^2 \\
&= \frac{1}{(G_T^{''})^2}\sum_{k=1}^{N}\left(\frac{2d_k\cos(\alpha-\phi_k)\sin(\alpha-\phi_k)}{b_k}\right)^2\sigma_d^2 \\
&\quad + \frac{1}{(G_T^{''})^2}\sum_{k=1}^{N}\left(-\frac{-2d_k^2\sin^2(\alpha-\phi_k)}{b_k}\right)^2\sigma_\phi^2 \\
&= \frac{1}{(G_T^{''})^2}\sum_{k=1}^{N}\left(\frac{4d_k^2\sin^2(\alpha-\phi_k)}{b_k^2}(\sigma_d^2\cos^2(\alpha-\phi_k) + \sigma_\phi^2 d_k^2\sin^2(\alpha-\phi_k))\right)
\end{aligned}$$

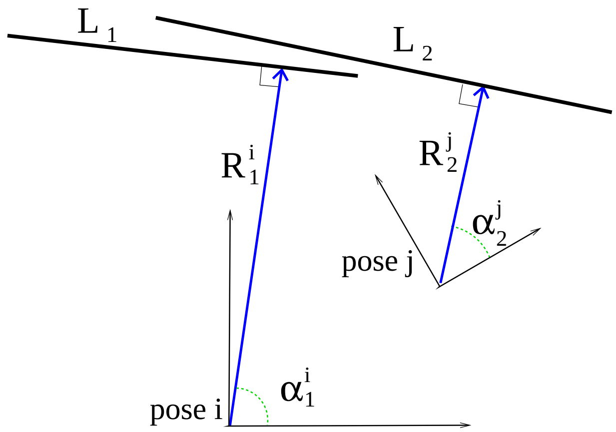
Use definition of b_k (Eq. (61)) to get

$$\begin{aligned}
P_{\alpha\alpha} &\simeq \frac{1}{(G_T^{''})^2}\sum_{k=1}^{N}\left(\frac{4d_k^2\sin^2(\alpha-\phi_k)}{b_k^2}(b_k)\right) \\
&= \frac{1}{(G_T^{''})^2}\sum_{k=1}^{N}\left(\frac{4d_k^2\sin^2(\alpha-\phi_k)}{b_k}\right) \\
&= \frac{1}{(G_T^{''})^2}\sum_{k=1}^{N}\left(\frac{4d_k^2\sin^2(\alpha-\phi_k)}{(\sigma_d^2\cos^2(\alpha-\phi_k) + \sigma_\phi^2 d_k^2\sin^2(\alpha-\phi_k))}\right)
\end{aligned}$$

4.3 Cross-correlation Covariance

$$
\begin{aligned}
P_{R\alpha} &= \sum_{k=1}^{N} \left(\nabla_{Y_k}^{T} g_\alpha\right) P_{Y_k Y_k} \left(\nabla_{Y_k} g_R\right) \\
&= \sum_{k=1}^{N} \left[\begin{array}{ccc} \nabla_{d_k} g_\alpha & | & \nabla_{\phi_k} g_\alpha \end{array}\right] \left[\begin{array}{cc} \sigma_d^2 & 0 \\ 0 & \sigma_\phi^2 \end{array}\right] \left[\begin{array}{c} \nabla_{d_k} g_R \\ - \\ \nabla_{\phi_k} g_R \end{array}\right] \\
&= \sum_{k=1}^{N} (\nabla_{d_k} g_\alpha) \sigma_d^2 (\nabla_{d_k} g_R) + \sum_{k=1}^{N} (\nabla_{\phi_k} g_\alpha) \sigma_\phi^2 (\nabla_{\phi_k} g_R) \\
&= \sum_{k=1}^{N} \left(\frac{-(\frac{a_d'}{b_k})}{G_T''}\right) \sigma_d^2 \left(\frac{\frac{\cos(\alpha-\phi_k)}{b_k}}{\sum_{j=1}^{N} \frac{1}{b_j}}\right) + \sum_{k=1}^{N} \left(\frac{-(\frac{a_\phi'}{b_k})}{G_T''}\right) \sigma_\phi^2 \left(\frac{\frac{d_k \sin(\alpha-\phi_k)}{b_k}}{\sum_{k=j}^{N} \frac{1}{b_j}}\right) \\
&= \frac{1}{G_T'' \left(\sum_{k=j}^{N} \frac{1}{b_j}\right)} \sum_{k=1}^{N} [\left(\frac{2 d_k \cos(\alpha-\phi_k) \sin(\alpha-\phi_k)}{b_k}\right) \sigma_d^2 \left(\frac{\cos(\alpha-\phi_k)}{b_k}\right) \\
&\quad + \left(\frac{2 d_k^2 \sin^2(\alpha-\phi_k)}{b_k}\right) \sigma_\phi^2 \left(\frac{d_k \sin(\alpha-\phi_k)}{b_k}\right)] \\
&= \frac{1}{G_T'' \left(\sum_{k=j}^{N} \frac{1}{b_j}\right)} \sum_{k=1}^{N} \left[\left(\frac{2 d_k \sin(\alpha-\phi_k)}{b_k^2}\right) \left(\sigma_d^2 \cos^2(\alpha-\phi_k) + \sigma_\phi^2 d_k^2 \sin^2(\alpha-\phi_k)\right)\right] \\
&= \frac{1}{G_T'' \left(\sum_{k=j}^{N} \frac{1}{b_j}\right)} \sum_{k=1}^{N} \left[\frac{2 d_k \sin(\alpha-\phi_k) b_k}{b_k^2}\right] \\
&= \frac{1}{G_T'' \left(\sum_{k=j}^{N} \frac{1}{b_j}\right)} \sum_{k=1}^{N} \left[\frac{2 d_k \sin(\alpha-\phi_k)}{b_k}\right]
\end{aligned}
$$

5 Transforming Across Reference Frames



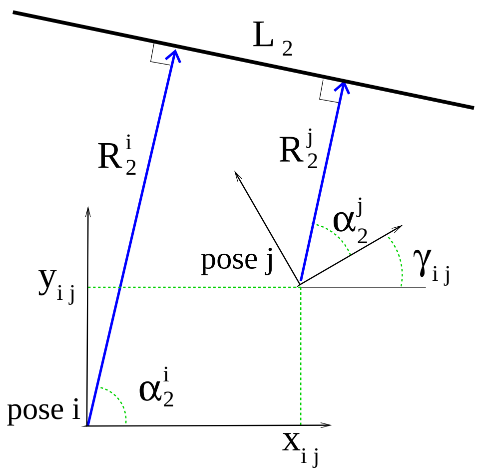
Consider L_1^i and L_2^j found in scans taken at poses i and j respectively.

$$L_1^i = \begin{bmatrix} R_1^j \\ \alpha_1^j \end{bmatrix} \quad L_2^j = \begin{bmatrix} R_2^j \\ \alpha_2^j \end{bmatrix} \tag{88}$$

Let $\hat{g}_{ij}$ be an independent measurement of the robot's pose j with respect to pose i with :

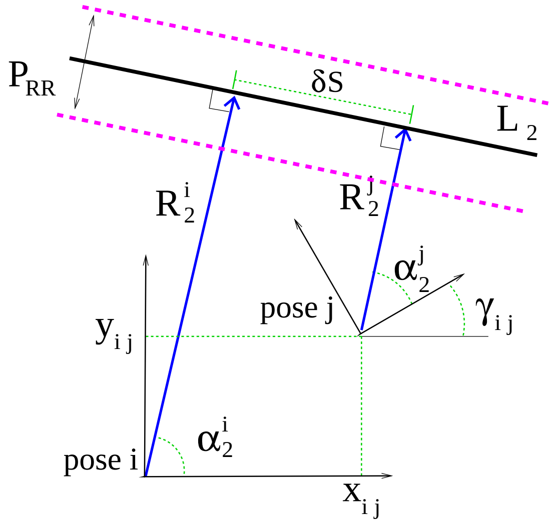
$$\hat{g}_{ij} = [x_{ij}, y_{ij}, \gamma_{ij}] \tag{89}$$

5.1 Line Coordinate Transformation



To transform the parameters of L_2 from pose j to pose i we calculate:

$$\begin{aligned} L_2^i &= \begin{bmatrix} R_2^i \\ \alpha_2^i \end{bmatrix} \\ &= \begin{bmatrix} R_2^j + x_{ij}\cos(\alpha_2^j + \gamma_{ij}) + y_{ij}\sin(\alpha_2^j + \gamma_{ij}) \\ \alpha_2^j + \gamma_{ij} \end{bmatrix} \end{aligned} \tag{90}$$



5.2 Line Covariance Transformation

Let the covariance of line L_2 with respect to pose j be represented as in Eq. (43) :

$$P_{L_2}^j = \begin{bmatrix} P_{RR} & P_{R\alpha} \\ P_{\alpha R} & P_{\alpha\alpha} \end{bmatrix} \tag{91}$$

In order to derive the proper reference frame transformations, we first augment Eq. (98)into a standard 3x3 position and orientation covariance matrix :

$$\widetilde{P}_{L_2}^j = \begin{bmatrix} P_{RR} & P_{RS} & P_{R\alpha} \\ P_{SR} & P_{SS} & P_{S\alpha} \\ P_{\alpha R} & P_{\alpha S} & P_{\alpha\alpha} \end{bmatrix} \tag{92}$$

with S is oriented orthogonal to the R direction and thus parallel to the line. We can then apply a standard reference frame transformation to $\widetilde{P}_{L_2}$ as follows:

$$\widetilde{P}_{L_2}^i = H\widetilde{P}_{L_2}^j H^T \tag{93}$$

with

$$H = \begin{bmatrix} 1 & 0 & \delta S \\ 0 & 1 & -\delta R \\ 0 & 0 & 1 \end{bmatrix} \tag{94}$$

and

$$\delta S = y_{ij}\cos(\alpha_2^i) - x_{ij}\sin(\alpha_2^i) \tag{95}$$

$$\delta R = x_{ij}\cos(\alpha_2^i) + y_{ij}\sin(\alpha_2^i) \tag{96}$$

Where δS is the pose displacement in the S direction and δR is the pose displacement in the R direction. Note that the transformation matrix H does not include a rotation because the orientation of the RS coordinate frame is fixed and defined by the line. It follows that :

$$\begin{aligned}
\widetilde{P}_{L_2}^i &= H\widetilde{P}_{L_2}^j H^T \\
&= \begin{bmatrix} 1 & 0 & \delta S \\ 0 & 1 & -\delta R \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_{RR} & P_{RS} & P_{R\alpha} \\ P_{SR} & P_{SS} & P_{S\alpha} \\ P_{\alpha R} & P_{\alpha S} & P_{\alpha\alpha} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \delta S & -\delta R & 1 \end{bmatrix} \\
&= \begin{bmatrix} P_{RR} + 2\delta S P_{R\alpha} + (\delta S)^2 P_{\alpha\alpha} & P_{RS} + P_{S\alpha}\delta S - \delta R P_{R\alpha} - \delta S\delta R P_{\alpha\alpha} & P_{R\alpha} + \delta S P_{\alpha\alpha} \\ P_{RS} + P_{S\alpha}\delta S - \delta R P_{R\alpha} - \delta S\delta R P_{\alpha\alpha} & P_{SS} - 2\delta R P_{S\alpha} + (\delta R)^2 P_{\alpha\alpha} & P_{S\alpha} - \delta R P_{\alpha\alpha} \\ P_{R\alpha} + \delta S P_{\alpha\alpha} & P_{S\alpha} - \delta R P_{\alpha\alpha} & P_{\alpha\alpha} \end{bmatrix}
\end{aligned}$$

But because we have no information in the S direction we can drop the terms which we had previously inserted and recover the transformed 2x2 covariance matrix as well as the corresponding transformation matrix B :

$$\begin{aligned} P^i_{L_2} &= \begin{bmatrix} P_{RR} + 2\delta S P_{R\alpha} + (\delta S)^2 P_{\alpha\alpha} & P_{R\alpha} + \delta S P_{\alpha\alpha} \\ P_{R\alpha} + \delta S P_{\alpha\alpha} & P_{\alpha\alpha} \end{bmatrix} \\ &= \begin{bmatrix} 1 & \delta S \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_{RR} & P_{R\alpha} \\ P_{\alpha R} & P_{\alpha\alpha} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \delta S & 1 \end{bmatrix} \end{aligned} \tag{97}$$

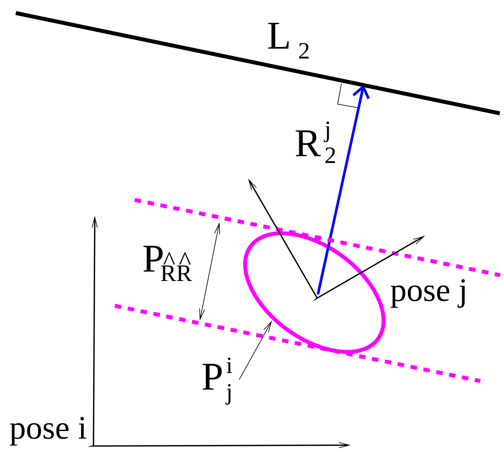
so

$$P^i_{L_2} = B P^j_{L_2} B^T \tag{98}$$

with

$$B = \begin{bmatrix} 1 & y_{ij}\cos(\alpha^i_2) - x_{ij}\sin(\alpha^i_2) \\ 0 & 1 \end{bmatrix} \tag{99}$$

5.3 Pose Covariance Transformation



The uncertainty in the pose displacement measurement $\hat{g}_{ij}$ (represented by covariance matrix P_{ij}) must be incorporated in the overall line covariance transformation. The line uncertainty from Eq. (98) can be combined with the uncertainty of the pose displacement measurement by projecting P_{ij} into the RS coordinate system. Let

$$P_{ij} = \begin{bmatrix} P_{xx} & P_{xy} & P_{x\gamma} \\ P_{yx} & P_{yy} & P_{y\gamma} \\ P_{\gamma x} & P_{\gamma y} & P_{\gamma\gamma} \end{bmatrix} \tag{100}$$

The matrix is rotated by $-\alpha^i_2$ to align with the RS reference frame and the terms corresponding to the S direction are dropped.

$$\begin{aligned} \widehat{P}^i_j &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(-\alpha^i_2) & -sin(-\alpha^i_2) & 0 \\ sin(-\alpha^i_2) & cos(-\alpha^i_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} P_{ij} \begin{bmatrix} cos(-\alpha^i_2) & sin(-\alpha^i_2) & 0 \\ -sin(-\alpha^i_2) & cos(-\alpha^i_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} cos(\alpha^i_2) & sin(\alpha^i_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} P_{ij} \begin{bmatrix} cos(\alpha^i_2) & 0 \\ sin(\alpha^i_2) & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} P_{xx}\cos^2(\alpha^i_2) + 2\cos(\alpha^i_2)\sin(\alpha^i_2)P_{xy} + \sin^2(\alpha^i_2)P_{yy} & P_{x\gamma}\cos(\alpha^i_2) + P_{y\gamma}\sin(\alpha^i_2) \\ P_{x\gamma}\cos(\alpha^i_2) + P_{y\gamma}\sin(\alpha^i_2) & P_{\gamma\gamma} \end{bmatrix} \\ &= \begin{bmatrix} P_{\widehat{R}\widehat{R}} & P_{\widehat{R}\gamma} \\ P_{\gamma\widehat{R}} & P_{\gamma\gamma} \end{bmatrix} \end{aligned} \tag{101}$$

so

$$\widehat{P}^i_j = KP_{ij}K^T \tag{102}$$

with

$$K = \begin{bmatrix} cos(\alpha^i_2) & sin(\alpha^i_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{103}$$

5.4 Combined Covariance Transformation

Therefore the covariance of line L_2 with respect to pose j can be transformed into the coordinate frame of pose i as follows:

$$\begin{aligned} {}^{tot}P^i_{L_2} &= P^j_{L_2} + \widehat{P}_{ij} \\ &= BP^j_{L_2}B^T + KP^{ij}K^T \end{aligned} \tag{104}$$

with B and K from Eqs. (99) and (103).

6 Line Merging

To determine whether a given pair of lines are sufficiently similar to warrant merging, we apply a criterion based on the chi-squared test. The coordinates and covariance matrices of the two lines as found by our line fitting algorithm are first represented with respect to a common pose i. We then apply the chi-squared test to determine if the difference between two lines is within the 3 sigma deviance threshold defined by the combined uncertainties of the lines. The merge criterion is

$$\chi^2 = (\delta L)^T (P^i_{L_1} + P^i_{L_2})^{-1} \delta L < 3 \tag{105}$$

with

$$\delta L = \begin{bmatrix} R^i_1 - R^i_2 \\ \alpha^i_1 - \alpha^i_2 \end{bmatrix}$$

We can derive the final merged line estimate using a maximum likelihood formulation with the necessary condition for most likely line L_m as follows:

$$\frac{\partial M}{\partial L_m} = 0 \tag{106}$$

$$M(L_m) = \sum_{k=1}^{N} (L_k - L_m)^T (P^L_k)^{-1} (L_k - L_m) \tag{107}$$

In order to minimize Eq. (107) we have to set

$$\begin{aligned} \frac{\partial M(L_m)}{\partial (L_m)} &= 0 \Leftrightarrow \\ \sum_{k=1}^{N} (P^L_k)^{-1} (L_k - L_m) &= 0 \Leftrightarrow \\ \sum_{k=1}^{N} (P^L_k)^{-1} (L_k) &= \left(\sum_{k=1}^{N} (P^L_k)^{-1} \right) L_m \Leftrightarrow \end{aligned}$$

$$L_m = \left(\sum_{k=1}^{N} (P^L_k)^{-1} \right)^{-1} \sum_{k=1}^{N} (P^L_k)^{-1} (L_k)$$

$$L^i_m = P^i_{L_m} \left((P^i_{L_1})^{-1} L^i_1 + (P^i_{L_2})^{-1} L^i_2 \right) \tag{108}$$

$$P^i_{L_m} = \left((P^i_{L_1})^{-1} + (P^i_{L_2})^{-1} \right)^{-1} \tag{109}$$