Poisson Process, Spike Train and All That

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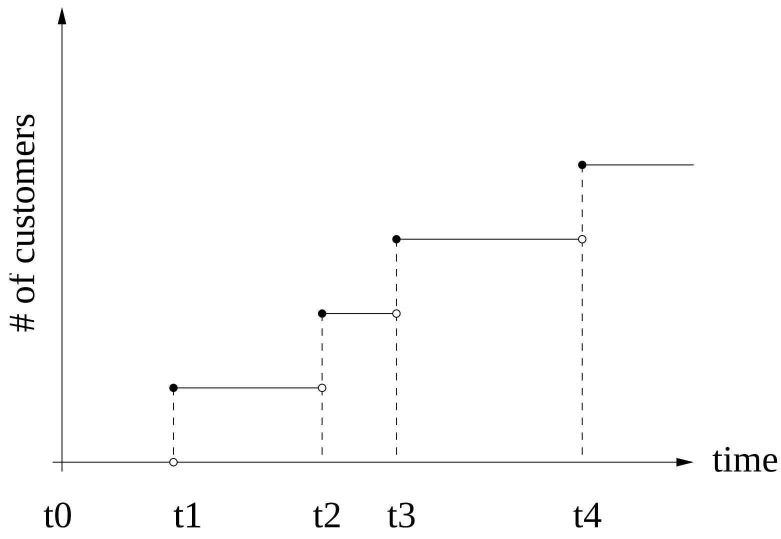


Figure 1: Counting process

1 Counting Process

Let Fig. 1 be a graph of the customers who are entering a bank. Every time a customer comes, the counter is increased by one. The time of the arrival of $i-$th customer is t_i. Since the customers are coming at random, the sequence $\{t_1, t_2, \cdots, t_m\}$, denoted shortly by $\{t_i\}$, is a random sequence. Also, the number of customers who came in the interval $(t_0, t]$ is a random variable (process). Such a process is right continuous, as indicated by the graph in Fig. 1.

As it is often case in the theory of stochastic processes, we assume that the **index set**, i.e. the set where $\{t_i\}$ is taking values from, is $T = [0, \infty)$. Therefore, we have a sequence of non-negative random variables

$$0 \leq t_0 < t_1 < t_2 < \cdots < t_m \to \infty \quad \text{as} \quad m \to \infty.$$

WLOG[1] let $t_0 = 0$ and $N_0 = 0$, then

$$N_t = \max\{n, \, t_n \leq t\}, \quad T = [0, \infty),$$

is called a **point process** (counting process), and is denoted shortly by $\{N_t, \, t \geq 0\}$.

[1]Without loss of generality

Let $T_n \triangleq t_n - t_{n-1}$ be inter-arrival time, then the sequence of inter-arrival times $\{T_n,\, n \geq 1\}$ is another stochastic process.

Special case is when $\{T_n,\, n \geq 1\}$ is a sequence of i.i.d.[2] random variables, then the sequence $\{t_n\}$ is called a **renewal process**. $\{N_t,\, t \geq 0\}$ is the associated renewal point process, sometimes also called renewal process. Also, keep in mind that $t_n = T_1 + T_2 + \cdots + T_n$.

Definition (Poisson process) A point process $\{N_t,\, t \geq 0\}$ is called a **Poisson process** if $N_0 = 0$ and $\{N_t\}$ satisfies the following conditions

1. its increments are stationary and its non-overlapping increments are independent
2. $P(N_{t+\triangle t} - N_t = 1) = \lambda\, \triangle t + o(\triangle t)$
3. $P(N_{t+\triangle t} - N_t \geq 2) = o(\triangle t)$

Remarks

- $\{N_t,\, t \in T\}$; $\quad t,\, s \in T$; $\quad t > s$; $\quad N_t - N_s$ - is the increment of stochastic process N_t.
- $N_{t+\triangle t} - N_t =$ the number of new arrivals during $(t,\, t + \triangle t]$.
- $\lambda = const > 0$ and $o(\triangle t)$ is understood as $\frac{o(\triangle t)}{\triangle t} \to 0$ when $\triangle t \to 0$.

  The Poisson process defined above is also known as **homogeneous Poisson process**. In general λ can be a time dependent function $\lambda(t)$, in which case we are dealing with **inhomogeneous Poisson process**. Finally, λ itself can be a realization of stochastic process $\lambda(t, \omega)$, in which case we have so-called **doubly stochastic Poisson process**.

  In any case, the parameter λ of a Poisson process is called the **rate** and sometimes the **intensity** of the process. Its dimension is [events]/[time] (e.g. spikes/sec in neuroscience).

Theorem Let $\{N_t,\, t \geq 0\}$ be a Poisson process, then

$$\boxed{P(N_t = k) = \frac{(\lambda\, t)^k}{k!} e^{-\lambda\, t}} \quad k = 0,\, 1,\, \cdots \tag{1}$$

[2] Independent identically distributed

The expression on the left hand side of (1) represents the probability of k arrivals in the interval $(0, t]$.

Proof A **generating function** of a discrete random variable X is defined via the following z-transform (recall that the moment generating function of a continuous random variable is defined through Laplace transform):

$$\mathbf{G}_X(z) = E[z^X] = \sum_{i=0}^{\infty} z^i \, p_i,$$

where $p_i = P(X = i)$. Let us assume that X is a Poisson random variable with parameter μ, then

$$P(X = i) = \frac{\mu^i}{i!} e^{-\mu} \qquad i = 0,\, 1,\, 2, \cdots$$

and

$$\mathbf{G}_X(z) = \sum_{i=0}^{\infty} z^i \, \frac{\mu^i}{i!} e^{-\mu} = e^{\mu(z-1)}. \tag{2}$$

Going back to Poisson process, define the generating function as

$$\mathbf{G}_t(z) \triangleq E[z^{N_t}]$$

Then we can write

$$\begin{aligned}\mathbf{G}_{t+\triangle t}(z) &= E[z^{N_{t+\triangle t}}] = E[z^{N_t + N_{t+\triangle t} - N_t}] = E[z^{N_t}] \, E[z^{N_{t+\triangle t} - N_t}] \\ &= \mathbf{G}_t(z) \, [(1 - \lambda \, \triangle t + o(\triangle t)) \, z^0 + (\lambda \, \triangle t + o(\triangle t)) \, z^1 + o(\triangle t)(z^2 + \cdots)]\end{aligned}$$

Furthermore

$$\begin{aligned}\frac{\mathbf{G}_{t+\triangle t} - \mathbf{G}_t(z)}{\triangle t} &= \mathbf{G}_t(z) \, [-\lambda + \frac{o(\triangle t)}{\triangle t} \; + \; (\lambda + \frac{o(\triangle t)}{\triangle t}) \, z + \frac{o(\triangle t)}{\triangle t}(z^2 + \cdots)] \\ &\Rightarrow \lim_{\triangle t \to 0} \frac{\mathbf{G}_{t+\triangle t} - \mathbf{G}_t(z)}{\triangle t} = \mathbf{G}_t(z) \, [-\lambda \; + \; \lambda \, z] \\ &\Rightarrow \frac{d\mathbf{G}_t(z)}{dt} = \mathbf{G}_z(t) \lambda \, (z - 1) \\ &\Rightarrow \log \mathbf{G}_t(z) - \underbrace{\log \mathbf{G}_0(z)}_{0} = \lambda \, t \, (z - 1) \\ &\Rightarrow \mathbf{G}_t(z) = e^{\lambda \, t \, (z-1)}\end{aligned}$$

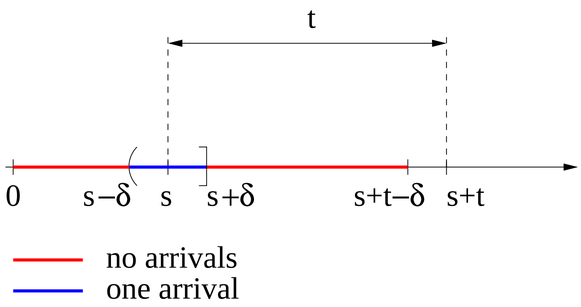


Figure 2: Event description

Comparing this result to (2) we conclude that N_t is a Poisson random variable with parameter $\lambda\, t$. ■

Theorem If $\{N_t,\, t \geq 0\}$ is a Poisson process and T_n is the inter-arrival time between the $n-$ th and $(n-1)-$ th events, then $\{T_n,\, n \geq 1\}$ is a sequence of i.i.d. random variables with exponential distribution, with parameter λ.

Proof

$$P(T_1 > t) = P(N_t = 0) = e^{-\lambda\, t} \quad \Rightarrow \quad T_1 - \text{exponential}$$

Need to show that T_1 and T_2 are independent and T_2 is also exponential.

$$P(T_2 > t \,|\, T_1 \in (s-\delta,\, s+\delta]) = \frac{P(T_2 > t,\, T_1 \in (s-\delta,\, s+\delta])}{P(T_1 \in (s-\delta,\, s+\delta])} \tag{3}$$

The event $\{T_2 > t,\, T_1 \in (s-\delta,\, s+\delta]\}$ is a subset of the event described by Fig. 2, i.e.

$$P(T_2 > t,\, T_1 \in (s{-}\delta,\, s{+}\delta]) \leq P(\underbrace{N_{s-\delta} = 0}_{\text{no arrivals}},\, \underbrace{N_{s+\delta} - N_{s-\delta} = 1}_{\text{one arrival}},\, \underbrace{N_{s+t-\delta} - N_{s+\delta} = 0}_{\text{no arrivals}})$$

$$P(T_2 > t,\, T_1 \in (s-\delta,\, s+\delta]) \leq P(T_1 \in (s-\delta,\, s+\delta])\, P(N_{s+t-\delta} - N_{s+\delta} = 0)$$
$$\Rightarrow P(T_2 > t,\, T_1 \in (s-\delta,\, s+\delta]) \leq P(T_1 \in (s-\delta,\, s+\delta])\, e^{-\lambda\,(t-2\,\delta)}$$

From (3) $\Rightarrow$

$$P(T_2 > t \,|\, T_1 \in (s-\delta,\, s+\delta]) \leq e^{-\lambda\,(t-2\,\delta)} \tag{4}$$

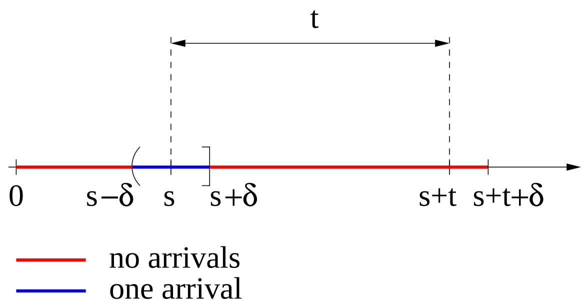


Figure 3: Event description

Similarly, the event described by Fig. 3 is a subset of the event $\{T_2 > t,\, T_1 \in (s-\delta,\, s+\delta]\}$, therefore

$$\begin{aligned} P(N_{s-\delta}=0,\, N_{s+\delta}-N_{s-\delta}=1,\, N_{s+t+\delta}-N_{s+\delta}=0) &\le P(T_2>t,\, T_1\in(s-\delta,\, s+\delta]) \\ \Rightarrow P(T_1\in(s-\delta,\, s+\delta])\, P(N_{s+t+\delta}-N_{s+\delta}=0) &\le P(T_2>t,\, T_1\in(s-\delta,\, s+\delta]) \\ \Rightarrow P(T_1\in(s-\delta,\, s+\delta])\, e^{-\lambda t} &\le P(T_2>t,\, T_1\in(s-\delta,\, s+\delta]) \end{aligned}$$

From (3) $\Rightarrow$

$$P(T_2 > t \,|\, T_1 \in (s-\delta,\, s+\delta]) \ge e^{-\lambda t} \tag{5}$$

From (4) and (5), using squeeze theorem ($\delta \to 0$), it follows

$$P(T_2 > t \,|\, T_1 = s) = e^{-\lambda t} \quad \Rightarrow \quad f_{T_2\,|\,T_1=s}(t\,|\,s) = \lambda\, e^{-\lambda t}$$

Therefore, T_2 is independent of T_1, and T_2 is exponentially distributed random variable. ■

Theorem

1. $E[N_t] = \lambda t$
2. $Var[N_t] = \lambda t$

Proof Recall that $\mathbf{G}_t(z) = E[z^{N_t}]$, then

$$\left[\frac{d\mathbf{G}_t(z)}{dz}\right]_{z=1} = E\left[N_t\, z^{N_t-1}\right]_{z=1} = E[N_t]$$
$$\Rightarrow E[N_t] = \left[\lambda\, t\, e^{\lambda\, t(z-1)}\right]_{z=1} = \lambda\, t$$

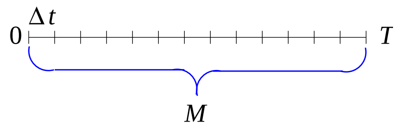


Figure 4: Uniform bins

Likewise

$$
\begin{aligned}
&\left[\frac{d^2\mathbf{G}_t(z)}{dz^2}\right]_{z=1} = E[N_t(N_t-1)] \\
&\Rightarrow E[N_t^2] = \left[(\lambda\, t)^2\, e^{\lambda\, t(z-1)}\right]_{z=1} + E[N_t] = (\lambda\, t)^2 + \lambda\, t \\
&\Rightarrow Var[N_t] = (\lambda\, t)^2 + \lambda\, t - (\lambda\, t)^2 = \lambda\, t \quad \blacksquare
\end{aligned}
$$

Theorem (Conditioning on the number of arrivals) Given that in the interval $(0,\, T]$ the number of arrivals is $N_T = n$, the n arrival times are independent and uniformly distributed on $[0,\, T]$.

Proof Independence of arrival times t_1, t_2 etc. directly follows from independence of non-overlapping increments. In particular let t_1 and t_2 be arrival times of first and second event, then

$$
\begin{aligned}
&P(t_1 \in (0,\, s],\, t_2 \in (s,\, t]) = P(N_s = 1,\, N_t - N_s = 1) = \\
&= P(N_s = 1)\, P(N_t - N_s = 1 | N_s = 1) = P(t_1 \in (0,\, s])\, P(t_2 \in (s,\, t])
\end{aligned}
$$

Suppose that we know exactly one event happened in the interval $(0,\, T]$, and suppose the interval is partitioned into M segments of length $\triangle t$, as shown in Fig. 1. Let p_i be the probability of event happening in the $i-$th bin, then $\sum_{i=1}^{M} p_i = 1$. From the definition of Poisson process it follows that $p_i \propto \lambda\, \triangle t$, say $p_i = C(\lambda\, \triangle\, t + o(\triangle t))$. The constant C is determined from

$$
\sum_{i=1}^{M} C(\lambda\, \triangle t \,+\, o(\triangle t)) = 1 \Rightarrow C = \frac{1}{\lambda\, M\, \triangle t + M\, o(\triangle t)} = \frac{1}{T(\lambda + \frac{o(\triangle t)}{\triangle t})}
$$

Let t_1 be a random variable corresponding to the time of arrival, then the **probability density function** (pdf) of t_1 can be defined as

$$
f_{t_1}(t) = \lim_{\triangle\, t \to 0} \frac{p_i}{\triangle t} = \frac{1}{T} \quad \forall i = 1, 2, \cdots, M \quad \text{where} \quad t = i\, \triangle t.
$$

Therefore, t_1 is uniformly distributed on $[0,\, T]$.
Let t_1 and t_2 be the arrival times of two events, and we know exactly two events happened on $(0,\, T]$. Also assume that t_1 and t_2 represent mere labels of events, not necessarily their order. Given that t_1 happened in $j-$th bin, the probability of t_2 occurring in any bin of size $\triangle t$ is proportional to the size of that bin, i.e. $p_i \propto \lambda \triangle t$, except for the $j-$th bin, where $p_j \propto o(\triangle t)$. By rendering the bin size infinitesimal, we notice that the probability p_i remains constant over all but one bin, the bin in which t_1 occurred, where $p_j = 0$. But this set is a **set of measure zero**, so the cumulative sum over p_i again gives rise to uniform distribution on $(0,\, T]$. ■

Question What is the probability of observing n events at instances $\tau_1,\, \tau_2,\, \cdots,\, \tau_n$ on the interval $[0,\, T]$?

Since arrival times $t_1,\, t_2\, \cdots,\, t_n$ are continuous random variables, the answer is 0. However, we can calculate the associated pdf as

$$f_{t_1\, t_2 \cdots t_n}(\tau_1,\, \tau_2,\, \cdots,\, \tau_n) = \\ = \lim_{dt \to 0} \frac{P(t_1 \in (\tau_1, \tau_1 + dt],\, \cdots,\, t_n \in (\tau_n, \tau_n + dt],\, N_T = n)}{dt^n}$$

where

$$P(t_1 \in (\tau_1, \tau_1 + dt],\, \cdots,\, t_n \in (\tau_n, \tau_n + dt],\, N_t = n) = \\ = P(t_1 \in (\tau_1, \tau_1 + dt],\, \cdots,\, t_n \in (\tau_n, \tau_n + dt] \,|\, N_T = n)\, P(N_T = n) \\ = \left(\frac{dt}{T}\right)^n \frac{(\lambda\, T)^n}{n!}\, e^{-\lambda\, T} = \frac{\lambda^n\, dt^n}{n!}\, e^{-\lambda\, T}$$

$$\Rightarrow f_{t_1\, t_2 \cdots t_n}(\tau_1,\, \tau_2,\, \cdots,\, \tau_n) = \frac{\lambda^n}{n!}\, e^{-\lambda\, T}$$

Question What is the **power spectrum** of Poisson process?

It does not make sense to talk about the power spectrum of Poisson process, since it is not a **stationary process**. In particular the mean of Poisson process is

$$E[N_t] = \lambda\, t$$

and its **autocorrelation function** is

$$R(t,s) \triangleq E[N_t\, N_s]$$

$$R(t,s) \overset{t \geq s}{=} E[(N_t - N_s + N_s)\, N_s] = E[(N_t - N_s)\, N_s + N_s^2]$$
$$= E[N_t - N_s]E[N_s] + E[N_s^2] = \lambda(t-s)\lambda\, s + \lambda^2\, s^2 + \lambda\, s = \lambda^2\, t\, s + \lambda\, s$$

$$R(t,s) \overset{t \leq s}{=} E[(N_s - N_t + N_t)\, N_t] = E[(N_s - N_t)\, N_t + N_t^2]$$
$$= E[N_s - N_t]E[N_t] + E[N_t^2] = \lambda(s-t)\lambda\, t + \lambda^2\, t^2 + \lambda\, t = \lambda^2\, t\, s + \lambda\, t$$

Since $R(t,s) \neq R(t-s)$, we conclude that $\{N_t,\, t \geq 0\}$ is not stationary (in weak sense), therefore it does not make sense to talk about its power spectrum. Let us define the following stochastic process (Fig. 5)

$$S_t = \frac{dN_t}{dt} = \sum_i \delta(t - t_i) \qquad \text{– spike train} \tag{6}$$

The fundamental lemma says that if $Y(t) = L\{X(t)\}$, where L is a linear

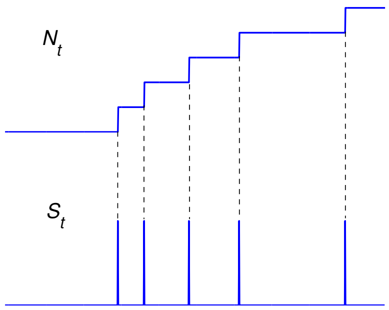


Figure 5: Spike train

operator, then

$$E[Y(t)] = L\{E[X(t)]\}$$

Since differentiation is a linear operator we have

$$E[S_t] = \frac{d(\lambda t)}{dt} = \lambda$$

Also, it can be shown using theory of linear operators that

$$R_{SS}(t,s) = \frac{\partial}{\partial t}\left[\frac{\partial R_{NN}(t,s)}{\partial s}\right] = \begin{cases} \frac{\partial}{\partial t}\left[\lambda^2 t + \lambda\right] & t > s \\ \frac{\partial}{\partial t}\left[\lambda^2 t\right] & t < s \end{cases}$$

$$= \frac{\partial}{\partial t}\left[\lambda^2 t + \lambda \underbrace{U(t-s)}_{\text{Heaviside function}}\right] = \lambda^2 + \lambda\,\delta(t-s)$$

Thus, S_t is WWS[3] stochastic process, and it makes sense to define the power spectrum of such a process as a Fourier transform of its autocorrelation function i.e.

$$P_S(\omega) = \mathcal{F}\{R_{SS}(\tau)\} = \int_{-\infty}^{\infty} R_{SS}(\tau)\, e^{-j\,\omega\,\tau} d\omega = \lambda + \lambda^2\, 2\pi\delta(\omega)$$

Therefore, the spike train $S_t = \sum_i \delta(t - t_i)$ of independent times t_i behaves almost as a **white noise**, since its power spectrum is flat for all frequencies, except for the spike at $\omega = 0$. The process S_t defined by (6) is a simple version of what is in engineering literature known as a **shot noise**.

Definition (Inhomogeneous Poisson process) A Poisson process with a non-constant rate $\lambda = \lambda(t)$ is called inhomogeneous Poisson process. In this case we have

1. non-overlapping increments are independent (the stationarity is lost though).
2. $P(N_{t+\triangle t} - N_t = 1) = \lambda(t)\,\triangle t + o(\triangle t)$
3. $P(N_{t+\triangle t} - N_t \geq 2) = o(\triangle t)$

Theorem If $\{N_t,\, t > 0\}$ is a Poisson process with the rate $\lambda(t)$, then N_t is a Poisson random variable with parameter $\mu = \int_0^t \lambda(\xi)\, d\xi$ i.e.

$$\boxed{P(N_t = k) = \frac{\left(\int_0^t \lambda(\xi)\, d\xi\right)^k}{k!}\, e^{-\int_0^t \lambda(\xi)\, d\xi}} \tag{7}$$

[3]Wide (weak) sense stationary. A stochastic process $X(t)$ is WSS if $E[X(t)] = const$ and $R_{XX}(t,s) = R_{XX}(t-s)$

Proof The proof of this theorem is identical to that of homogeneous case except that λ is replaced by $\lambda(t)$. In particular, one can easily get

$$\mathbf{G}_t(z) = e^{(z-1)\int_0^t \lambda(\xi)\,d\xi}, \tag{8}$$

from which (7) readily follows. ■

Theorem Let $\{N_t,\, t > 0\}$ be an inhomogeneous Poisson process with the rate $\lambda(t)$ and let $t > s \geq 0$, then

$$\boxed{P(N_t - N_s = k) = \frac{(\int_s^t \lambda(\xi)\,d\xi)^k}{k!}\, e^{-\int_s^t \lambda(\xi)\,d\xi}} \tag{9}$$

The application of this theorem stems from the fact that we cannot use $P(N_t - N_s = k) = P(N_{t-s} = k)$, since the increments are no longer stationary.

Proof

$$\mathbf{G}_t(z) = E[z^{N_t}] = E[z^{N_t - N_s + N_s}] = E[z^{N_t - N_s}]\, E[z^{N_s}] = E[z^{N_t - N_s}]\, \mathbf{G}_s(z)$$
$$\Rightarrow E[z^{N_t - N_s}] = \frac{\mathbf{G}_t(z)}{\mathbf{G}_s(z)} \overset{\text{by }(8)}{=} \frac{e^{(z-1)\int_0^t \lambda(\xi)\,d\xi}}{e^{(z-1)\int_0^s \lambda(\xi)\,d\xi}} = e^{(z-1)\int_s^t \lambda(\xi)\,d\xi}$$

Thus, $N_t - N_s$ is a Poisson random variable with parameter $\mu = \int_s^t \lambda(\xi)\,d\xi$, and (9) easily follows. ■

Theorem

1. $E[N_t] = \int_0^t \lambda(\xi)\,d\xi$
2. $Var[N_t] = \int_0^t \lambda(\xi)\,d\xi$

Proof Recall that

$$E[N_t] = \left[\frac{dG_z(t)}{dz}\right]_{z=1} \quad \text{and} \quad E[N_t^2] = \left[\frac{d^2 G_z(t)}{dz^2}\right]_{z=1} + E[N_t]$$

From (8) we have $\mathbf{G}_t(z) = e^{(z-1)\int_0^t \lambda(\xi)\,d\xi}$, and the two results follow after immediate calculations. ■

Theorem (Conditioning on the number of arrivals) Given that in the interval $(0, T]$ the number of arrivals is $N_T = n$, the n arrival times are independently distributed on $[0, T]$ with the pdf $\lambda(t)/\int_0^T \lambda(\xi)\, d\xi$.

Proof The proof of this theorem is analogous to that of the homogeneous case. The probability of a single event happening at any of M bins (Fig. 1) is given by $p_i = C(\lambda(i\,\triangle t)\,\triangle t + o(\triangle t))$, where i is the bin index. Given that exactly one event occurred in the interval $(0, T]$, we have

$$\sum_{i=1}^{M} p_i = 1 \;\Rightarrow\; C = \frac{1}{\sum_{i=1}^{M} \lambda(i\triangle t)\,\triangle t + T\,\frac{o(\triangle t)}{\triangle t}}$$

$$f_{t_1}(t) = \lim_{\triangle t \to \infty} \frac{p_i}{\triangle t} = \frac{\lambda(t)}{\int_0^T \lambda(\xi)\, d\xi} \quad \text{where} \quad t = i\,\triangle t.$$

The argument for independence of two or more arrival times is identical to that of the homogeneous case. ■

Question What is the probability of observing n events at instances τ_1, τ_2, $\cdots$, τ_n on the interval $[0, T]$?

Since arrival times t_1, t_2 $\cdots$, t_n are continuous random variables, the answer is 0. However, we can calculate the associated pdf as

$$f_{t_1\, t_2 \cdots t_n}(\tau_1,\, \tau_2,\, \cdots,\, \tau_n) = $$
$$= \lim_{dt \to 0} \frac{P(t_1 \in (\tau_1, \tau_1 + dt],\, \cdots,\, t_n \in (\tau_n, \tau_n + dt],\, N_T = n)}{dt^n}$$

where

$$P(t_1 \in (\tau_1, \tau_1 + dt],\, \cdots,\, t_n \in (\tau_n, \tau_n + dt],\, N_T = n) =$$
$$= P(t_1 \in (\tau_1, \tau_1 + dt],\, \cdots,\, t_n \in (\tau_n, \tau_n + dt] \,|\, N_T = n)\, P(N_T = n)$$
$$= \left[\prod_{i=1}^{n} \frac{\int_{\tau_i}^{\tau_i + dt} \lambda(\tau)\, d\tau}{\int_0^T \lambda(\xi)\, d\xi}\right] \frac{(\int_0^T \lambda(\xi)\, d\xi)^n}{n!}\, e^{-\int_0^T \lambda(\xi)\, d\xi}$$
$$\overset{dt \to 0}{\approx} \left[\prod_{i=1}^{n} \lambda(\tau_i)\right] \frac{dt^n}{n!}\, e^{-\int_0^T \lambda(\xi)\, d\xi}$$
$$\Rightarrow f_{t_1\, t_2 \cdots t_n}(\tau_1,\, \tau_2,\, \cdots,\, \tau_n) = \frac{\prod_{i=1}^{n} \lambda(\tau_i)}{n!}\, e^{-\int_0^T \lambda(\xi)\, d\xi}$$

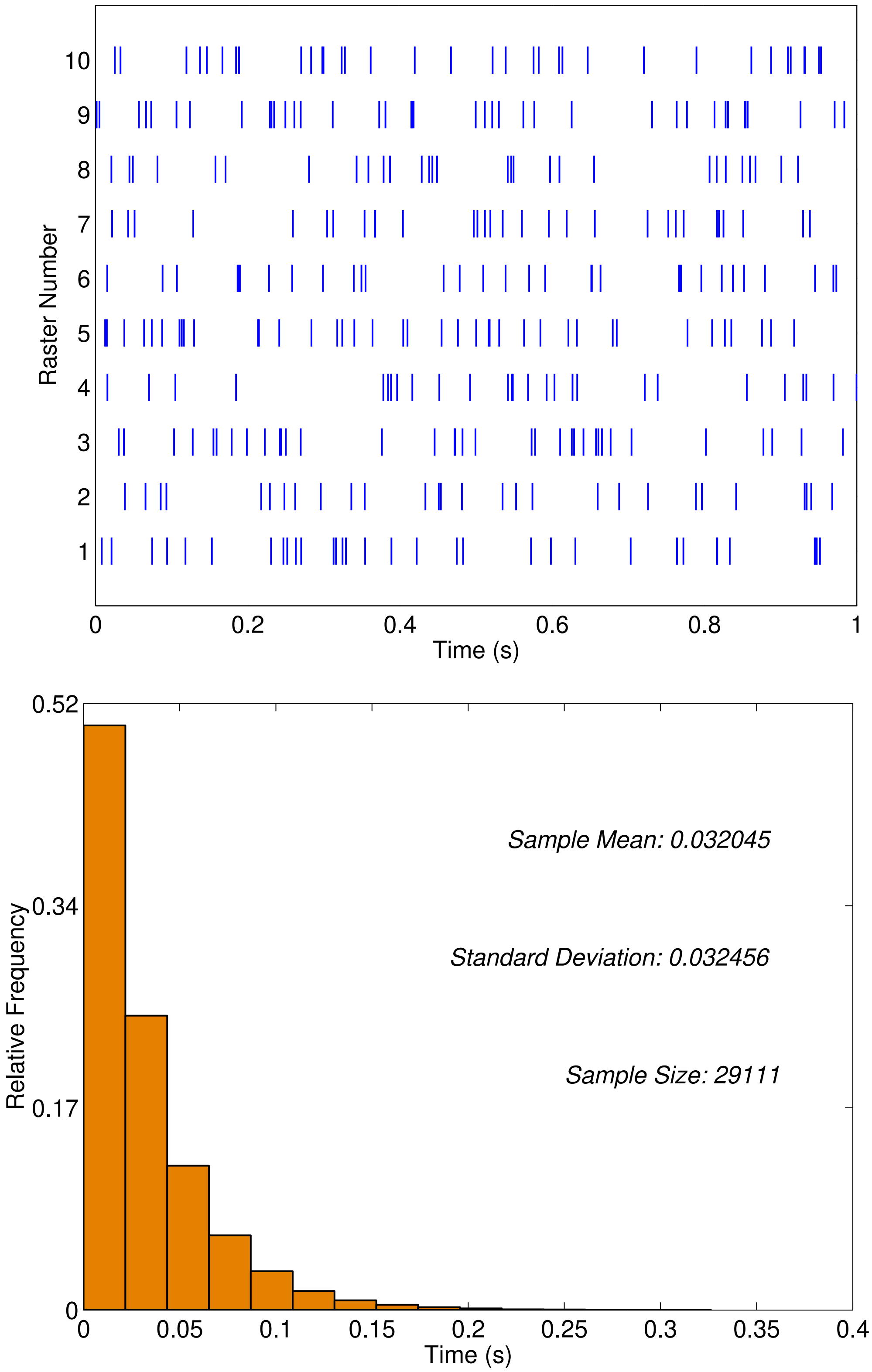


Figure 6: Realization of a point process using conditioning on the number of arrivals. (Top) Ten different sample paths of the same point process shown as raster plots. (Bottom) The histogram of inter-arrival times, showing the exponential trend

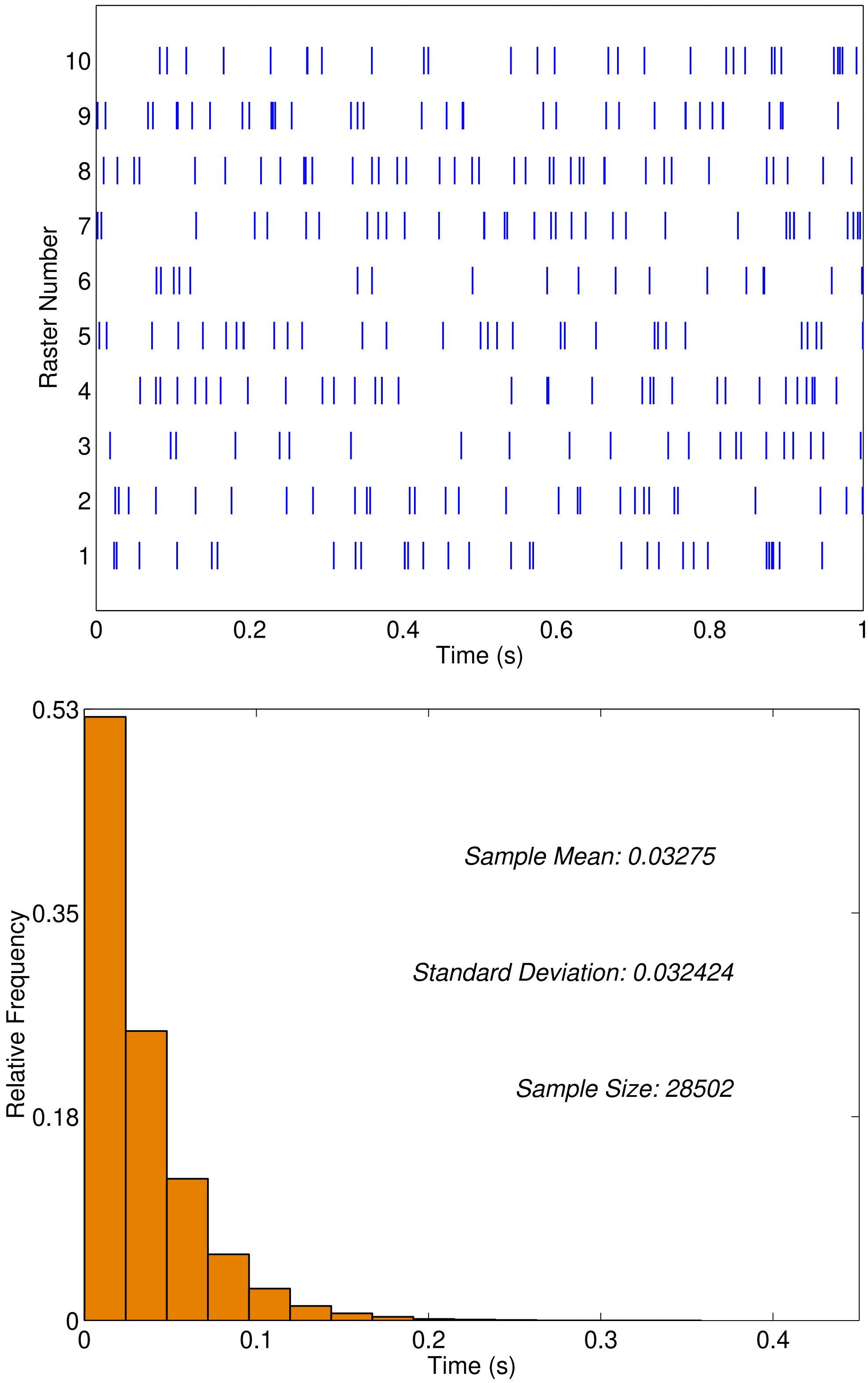
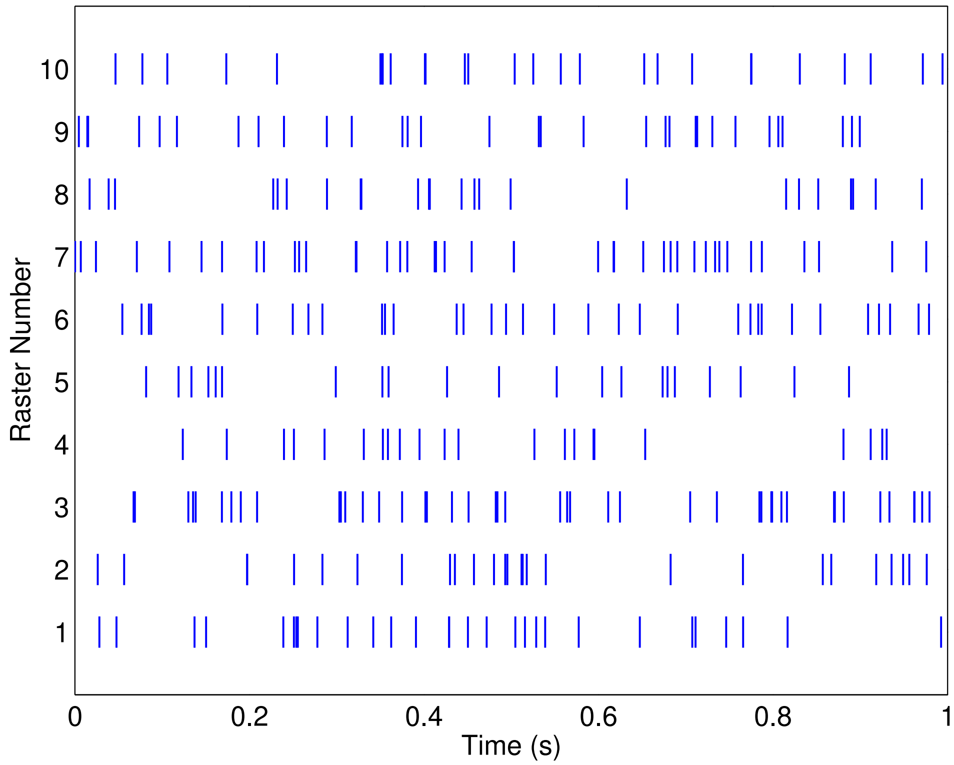


Figure 7: Realization of a point process using method of infinitesimal increments. (Top) Ten different sample paths of the same point process shown as raster plots. (Bottom) The histogram of inter-arrival times, showing the exponential trend



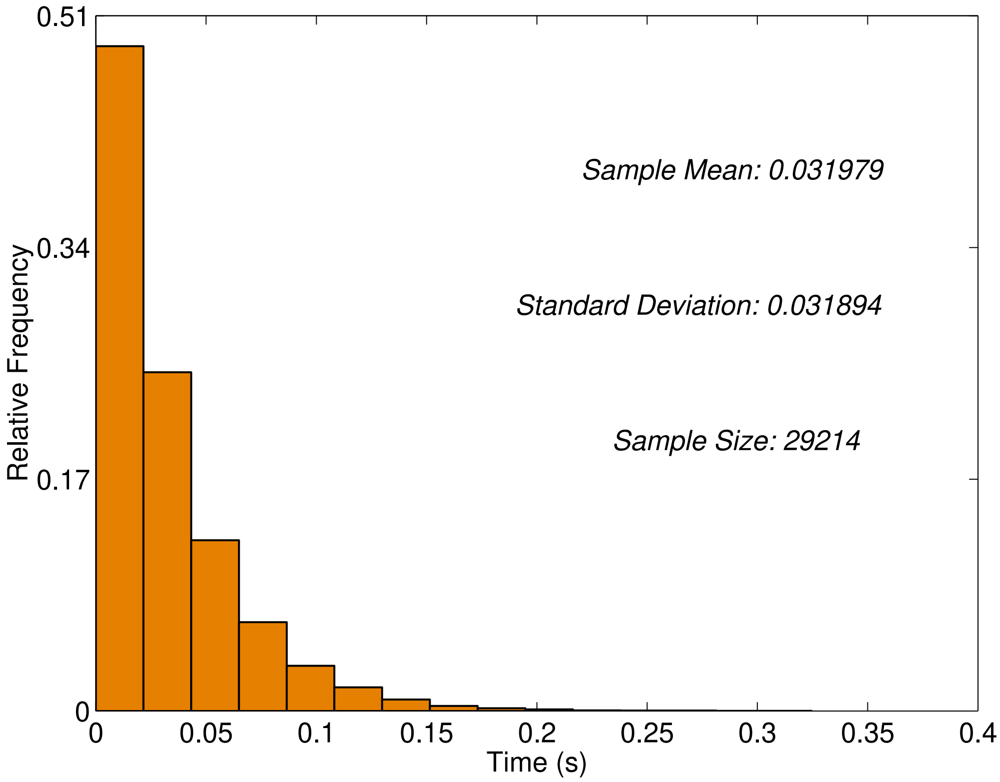


Figure 8: Realization of a point process using method of independent inter-arrival times. (Top) Ten different sample paths of the same point process shown as raster plots. (Bottom) The histogram of inter-arrival times, showing the exponential trend