Problem #1:

Recall that the definition of the evolute, \( \beta(t) \), of a planar curve, \( \alpha(t) \), is:

\[
\beta(t) = \alpha(t) + \frac{1}{\kappa(t)} \vec{n}(t)
\]

where \( \kappa(t) \) is the curvature of \( \alpha(t) \) and \( \vec{n}(t) \) is the unit normal vector of \( \alpha(t) \) at \( t \). Let’s assume that the curve is arc-length parametrized, i.e., \( t \) is the arc-length parameter.

The tangent to the evolute is simply derived by taking the derivative of the evolute equation.

\[
\frac{d\beta}{dt} = \vec{u} + \frac{d}{dt} \left( \frac{1}{\kappa(t)} \vec{n}(t) \right) = \vec{u} + \frac{\kappa(t) \vec{n}'(t) - \vec{n}(t) \kappa'(t)}{\kappa^2(t)}
\]

Recall that for a regular curve, \( \vec{u}(t) = \alpha'(t) \), \( \vec{u}'(t) = \kappa(t) \vec{n}(t) \).

What is \( \vec{n}'(t) \) for a planar curve? Since \( \vec{n}(t) \) is a unit length curve, then \( \vec{n}'(t) \) must be a vector orthogonal to \( \vec{n}(t) \)–i.e., a vector in the direction of \( \vec{u}(t) \). Thus, assume that \( \vec{n}' = \gamma \vec{u} \), where \( \gamma \) is some proportionality constant, which is to be determined.

From the relationship \( \vec{u} \cdot \vec{n} = 0 \), we can obtain (by taking derivatives):

\[
0 = \vec{u}' \cdot \vec{n} + \vec{u} \cdot \vec{n}' = \kappa + \vec{u} \cdot \vec{n}' = \kappa + \vec{u} \cdot (\gamma \vec{u}) = \kappa + \gamma
\]

This implies that:

\( \vec{n}' = -\kappa \vec{u} \)

Combining these results, gives:

\[
\frac{d\beta(t)}{dt} = \vec{u} + \frac{\kappa(t) \vec{n}' - \vec{n}(t) \kappa'}{\kappa^2} = \vec{u} + \frac{-\kappa^2 \vec{u} - \kappa' \vec{n}}{\kappa^2} = -\frac{\kappa'}{\kappa} \vec{n}'
\]

Thus, \( d\beta/dt \) is a vector in the direction of the normal to \( \alpha(t) \).

Part (b): The evolute of a circle is simply a point.
For a circle:
\[\alpha(t) = \begin{bmatrix} r \cos(\frac{t}{r}) \\ r \sin(\frac{t}{r}) \end{bmatrix}\]  
(9)
\[\beta(t) = \begin{bmatrix} r \cos(\frac{t}{r}) \\ r \sin(\frac{t}{r}) \end{bmatrix} + r \begin{bmatrix} -\cos(\frac{t}{r}) \\ \sin(\frac{t}{r}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\]  
(10)

**Problem #2**

**Part (a):** Let’s define the object frames on the two bodies as follows:

Recall that the boundary of the ellipse can thus be parametrized as:
\[c_1(\mu_1) = \begin{bmatrix} a \cos(\mu_1) \\ b \sin(\mu_1) \end{bmatrix}\]

where \(\mu_1\) is the “curve parameter” of the ellipse boundary. It is not necessarily the arc-length parameter. Note, with this parametrization, the normal vector (see below) is pointing inward into the object. The circle can be parameterized as:
\[c_2(\mu_2) = \begin{bmatrix} r \cos(\mu_2) \\ -r \sin(\mu_2) \end{bmatrix}\]

Thus, the tangent vectors (not necessarily unit length) to the two surfaces are:
\[\vec{t}_1(\mu_1) = \begin{bmatrix} -a \sin(\mu_1) \\ b \cos(\mu_1) \end{bmatrix}\]  
(11)
\[\vec{t}_2(\mu_2) = \begin{bmatrix} -r \sin(\mu_2) \\ -r \cos(\mu_2) \end{bmatrix}\]  
(12)

Similarly:
\[M_1 = |\vec{t}_1| = (a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{1/2}\]  
(13)
\[M_2 = |\vec{t}_2| = r\]  
(14)

Recall that the curvature of the \(i^{th}\) planar surface could be computed as:
\[\kappa_i = -M_i^{-2} (\partial c_i / \mu_i)^T \frac{\partial \vec{n}_i}{\partial \mu_i}\]

For planar curves, the normal can be easily computed as the unit tangent vector rotated by \(\pm \pi/2\)
\[\vec{n}_1 = -\frac{1}{(a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{1/2}} \begin{bmatrix} b \cos \mu_1 \\ a \sin \mu_1 \end{bmatrix}\]
Consequently:

\[
\frac{\partial \vec{n}_i}{\partial \mu_1} = -\frac{ab}{(a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{3/2}} \begin{bmatrix} -a \sin \mu_1 \\ b \cos \mu_1 \end{bmatrix}
\]

Thus:

\[
\kappa_1 = \frac{ab}{(a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{3/2}}
\]

\[
\vec{n}_2 = \begin{bmatrix} -r \cos(\mu_2) \\ -r \sin(\mu_2) \end{bmatrix}
\]

Consequently:

\[
\frac{\partial \vec{n}_2}{\partial \mu_2} = R
\]

Thus:

\[
\kappa_2 = \frac{1}{r}
\]

Assuming that \(v_t\) and \(\dot{\theta}\) are given, the contact equations are:

\[
\dot{\mu}_1 = (\kappa_1 + \kappa_2)^{-1} M_1^{-1} (-\dot{\theta} + \kappa_2 v_t) \tag{15}
\]

\[
= \left( \frac{ab}{(a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{3/2}} + \frac{1}{r} \right)^{-1} (a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{1/2} \left(-\dot{\theta} + \frac{1}{r} v_t\right) \tag{16}
\]

\[
\dot{\mu}_2 = (\kappa_1 + \kappa_2)^{-1} M_2^{-1} (\dot{\theta} + \kappa_1 v_t) \tag{17}
\]

\[
= \left( \frac{ab}{(a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{3/2}} + \frac{1}{r} \right)^{-1} \frac{1}{r} (\dot{\theta} + \frac{ab}{(a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{3/2}} v_t) \tag{18}
\]

The relative curvature is ill defined when \(\kappa_1 + \kappa_2 = 0\). In other words, when:

\[
\frac{ab}{(a^2 \sin^2 \mu_1 + b^2 \cos^2 \mu_1)^{3/2}} = -\frac{1}{r} \tag{19}
\]