Review of Manifolds, Lie Groups, and Lie Algebras

Joel W. Burdick and Patricio Vela

California Institute of Technology
Mechanical Engineering, BioEngineering
Pasadena, CA 91125, USA
Manifolds

Systems evolve on a manifold, $Q$.

**Definition 1** Let $X, Y$ be subsets of two Euclidean spaces and let $f: X \to Y$ be bijective. If $f$ and $f^{-1}$ are continuous, then $f$ is a homeomorphism. If $f$ and $f^{-1}$ are smooth, then $f$ is a diffeomorphism.

**Definition 2** A $k$-dimensional manifold, $M$, is locally diffeomorphic to $\mathbb{R}^k$. I.e., for each $x \in M$, there exists a nbhd of $x$, $V \subset M$, which is diffeomorphic to an open set $U \subset \mathbb{R}^k$. 
**Definition 3** A coordinatizable surface, \( S \), is the image of a map \( f: U \rightarrow \mathbb{R}^3 \) where

- \( U \) is an open connected subset of \( \mathbb{R}^2 \).
- The vectors \( \frac{\partial f}{\partial u} \) and \( \frac{\partial f}{\partial v} \) are linearly independent for all \((u, v) \in U\).
- \( f \) is a homeomorphism.

\((f, U)\) is a coordinate system for \( S \) with coordinates \( u, v \).

\( f^{-1} \) is termed a local parametrization.
Example (Unit Sphere)

One coordinate system for the sphere is:

\[ U = \{(u, v) \mid -\frac{\pi}{2} < u < \frac{\pi}{2}; -\pi < v < \pi\} \]

\[ f(u, v) = \begin{bmatrix} \cos(u) \cos(v) \\ -\cos(u) \sin(v) \\ \sin(u) \end{bmatrix} \]

Note that \( \frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v} = 0 \), implying that \((f, U)\) is an orthogonal coordinate system.
Tangent Spaces, Vectors

Definition 4  The tangent space to $M$ at $x \in M$, denoted by $T_x M$, is the image of $df \mid_{f^{-1}(x)}$

Remarks:

1. $T_p S$, is the closest linear approximation to $M$ at $p$.
2. Generally, if $p_1 \neq p_2$, then $T_{p_1} S \neq T_{p_2} S$.
3. The \textit{dimension} of a manifold, $M$, is defined as the dimension of its tangent space: $\dim(M) = \dim(T_p M)$.
4. The definition of the tangent space is intrinsic.

- A \textit{tangent vector} at $p \in M$ is a vector in $T_p M$.
- The union of all tangent spaces is the \textit{tangent bundle}. 
Example (sphere continued)

Let $p = f(0, 0) = [1 \ 0 \ 0]^T$ (where the $x$-axis intersects the sphere’s surface). Then

$$df_{(0,0)} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Therefore, $T_p M$ is the plane passing through $p$ and parallel to the $y$-$z$ plane.
**Definition 5**  The manifold $\mathcal{B}$ is a fiber bundle if the following exist:

1. a manifold $M$ called the base space,
2. a projection $\pi : \mathcal{B} \to M$, and
3. a space $Y$ called the fiber.

The set $Y_x$, defined by

$$Y_x = \pi^{-1}(x)$$

is called the fiber over the point $x$ of $M$. Each $Y_x$ is homeomorphic to $Y$.

$\mathcal{B}$ is a vector bundle if $Y$ is a vector space.
Vector Fields

A vector field is a section defined on the tangent bundle $TQ$, denoted

$$X : Q \rightarrow TQ$$

For each element $q \in Q$, $X(q) \in T_qQ$.

If the vector field is time dependent, then it is written $X(q, t)$ with shorthand notation,

$$X_t(\cdot) \equiv X(\cdot, t)$$
The Jacobi-Lie Bracket

The Jacobi-Lie bracket is defined as,

\[ [X, Y] = L_X Y \equiv \lim_{t \to 0} \frac{1}{t} \left( \left( \Phi_t^X \right)^* Y(q) - Y(q) \right) \]

where, \( \Phi_t^X \) is the flow of the vector field \( X \).

The Jacobi-Lie bracket is used to characterize:

1. Involutivity (closure of bracket).
2. Flows (noncommutativity).
The Jacobi-Lie Bracket

The Jacobi-Lie bracket is defined as,

\[ [X, Y] = L_X Y \equiv \lim_{t \to 0} \frac{1}{t} \left( (\Phi^X_t)^* Y(q) - Y(q) \right) \]

where, \( \Phi^X_t \) is the flow of the vector field \( X \).

The Jacobi-Lie bracket is used to characterize:

1. Involutivity (closure of bracket).
2. Flows (noncommutativity).

\[ [X, Y] \in \Delta, \quad \forall X, Y \in \Delta \]
The Jacobi-Lie Bracket

The Jacobi-Lie bracket is defined as,

\[ [X, Y] = L_X Y \equiv \lim_{t \to 0} \frac{1}{t} \left( \left( \Phi_t^X \right)^* Y(q) - Y(q) \right) \]

where, \( \Phi_t^X \) is the flow of the vector field \( X \).

The Jacobi-Lie bracket is used to characterize:

1. Involutivity (closure of bracket).
2. Flows (noncommutativity).

\[ [X, Y] \neq 0 \]
The Jacobi-Lie Bracket

The Jacobi-Lie bracket is defined as,

$$[X, Y] = L_X Y \equiv \lim_{t \to 0} \frac{1}{t} \left( \left( \Phi^X_t \right)^* Y(q) - Y(q) \right)$$

where, $\Phi^X_t$ is the flow of the vector field $X$.

The Jacobi-Lie bracket is used to characterize:

1. Involutivity (closure of bracket).
2. Flows (noncommutativity).

$$[X(z), Y(z)] = \frac{\partial Y}{\partial z} X - \frac{\partial X}{\partial z} Y$$
Lie Groups
Definition 5  A group is a nonempty set $G$ with a product operation, $\ast$, such that the following hold:

1. Associativity Law: $a \ast (b \ast c) = (a \ast b) \ast c$.
2. Closed Operation: $a \ast b \in G$ if $a, b \in G$.
3. Identity: $e \ast x = x \ast e = x$.
4. Inverse: $\forall x \in G$, $\exists y : x \ast y = y \ast x = e$.

Definition 5  A Lie group is a manifold $G$ whose group structure is consistent with its manifold structure. I.e., group multiplication,

$$\mu : G \times G \rightarrow G, \quad (g, h) \mapsto gh,$$

is $C^\infty$, as is inversion.
The Classical Matrix Groups

Definition 5  The set of $n \times n$ invertible matrices under matrix multiplication forms group, denoted by $GL(n)$.

Definition 5  A subset, $H \subset G$, is a subgroup of $G$, if $H$ is itself a group under the operation of $G$.

Some of the classical subgroups of $GL(n)$:

1. $SL(n)$: $n \times n$ matrices with $\det = +1$
2. $O(n)$: $n \times n$ orthogonal matrices ($A^T A = I$)
3. $SO(n)$: $n \times n$ in both $SL(n)$ and $O(n)$
4. $U(n)$: $n \times n$ complex orthogonal matrices
5. $SU(n)$: matrices in $U(n)$ with $\det = +1$. 
Actions of Lie Groups

The product structure can be used to define a *left translation*,

\[ L_g : G \to G, \quad L_g(h) = gh, \]

and similarly a *right translation*,

\[ R_g : G \to G, \quad R_g(h) = hg. \]

Note that,

\[ L_{g_1} \circ L_{g_2} = L_{g_1 g_2} \quad \text{and} \quad R_{g_1} \circ R_{g_2} = R_{g_2 g_1}. \]

An *inner automorphism* may be defined,

\[ I_g : G \to G, \quad I_g(h) = L_g R_{g^{-1}}(h) = R_{g^{-1}} L_g(h) = ghg^{-1} \]
Lie Group: $SO(3)$

$SO(3)$ is the group of rotations in Euclidean space, $\mathbb{R}^3$. As a matrix Lie group, $g \in SO(3)$ satisfies:

- $gg^T = I$.
- $\det(g) = 1$. 

![Diagram of rotations in 3D space]
\(SO(3)\) is the group of rotations in Euclidean space, \(\mathbb{R}^3\). As a matrix Lie group, \(g \in SO(3)\) satisfies:

- \(gg^T = I\).
- \(\det(g) = 1\).
Lie Group: $SE(2)$

$SE(2)$ describes rigid body motions in the Euclidean plane. As a matrix Lie group, $g \in SE(2)$ takes the form:

$$
g = \begin{bmatrix}
\cos \theta & -\sin \theta & x \\
\sin \theta & \cos \theta & y \\
0 & 0 & 1
\end{bmatrix}
$$
\textbf{Lie Group: } $SE(2)$

$SE(2)$ describes rigid body motions in the Euclidean plane. As a matrix Lie group, $g \in SE(2)$ takes the form:

$$g = \begin{bmatrix} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{bmatrix}$$
Lie Group: $\text{Diff}_{vol}(M)$

$\text{Diff}_{vol}(M)$ is the Lie group of volume preserving diffeomorphisms of a manifold $M$.

An element $g \in \text{Diff}_{vol}(M)$ is a mapping $g : M \rightarrow M$.
Lie Group: $\text{Diff}_{\text{vol}}(M)$

$\text{Diff}_{\text{vol}}(M)$ is the Lie group of volume preserving diffeomorphisms of a manifold $M$.

An element $g \in \text{Diff}_{\text{vol}}(M)$ is a mapping

$$g : M \rightarrow M$$
Invariant Vector Fields

A vector field $X$ on $G$ is left-invariant if

$$(T_h L_g) X(h) = X(gh)$$

The set of left-invariant vector-fields, $\mathcal{X}_L(G)$, form a Lie sub-algebra since,

$$L_g^*[X, Y] = [L_g^*X, L_g^*Y] = [X, Y]$$
Elements in $\mathcal{X}_L(G)$ can be identified with $T_eG$.

$$X(g) = X_\xi(g) = T_e L_g \xi$$

The Jacobi-Lie bracket defined at the point $e \in G$,

$$[\xi, \eta] = [X_\xi, X_\eta](e)$$

gives the tangent space $TeG$ a bracket structure.

This bracket is called the *Lie bracket*, and makes $TeG$, denoted by $g$, into a Lie algebra.
**Notes on Lie Algebras**

**Lie Algebra:** A real vector space, $V$, with a multiplication operation $[ , ]$ which satisfies for $A, B \in V$:

1. $[A, B] = -[B, A];$
2. $[A, B + C] = [A, B] + [A, C]; 
   \quad [A + B, C] = [A, C] + [B, C];$
3. for $r \in \mathbb{R}, \quad r[A, B] = [rA, B] = [A, rB]$

The set of smooth vectors fields on a manifold $M$ forms a Lie Algebra under Jacobi-Lie bracket operation.
Examples

Lie Algebra of $GL(n, \mathbb{R})$: Set of all $n \times n$ real matrices

Lie Algebra of $SO(3)$: Set of $3 \times 3$ skew-symmetric matrices, denoted by $so(3)$:

$$\hat{\omega} = \begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix}$$

The Lie Bracket is the matrix commutator:

$$\hat{\omega}_1, \hat{\omega}_2 = \hat{\omega}_1 \hat{\omega}_2 - \hat{\omega}_2 \hat{\omega}_1$$
Examples Continued

Lie Algebra of $SE(3)$: Matrices in $se(3)$ take the form:

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & \vec{v} \\ 0^T & 1 \end{bmatrix}; \quad \hat{\omega} \in so(3); \quad \vec{v} \in \mathbb{R}^3$$

The Lie Bracket is given by:

$$[\hat{\xi}_1, \hat{\xi}_2] = \hat{\xi}_1 \hat{\xi}_2 - \hat{\xi}_2 \hat{\xi}_1 = \begin{bmatrix} [\hat{\omega}_1, \hat{\omega}_2] & \hat{\omega}_1 \vec{v}_2 - \hat{\omega}_2 \vec{v}_1 \\ 0^T & 0 \end{bmatrix}$$
The Adjoint

Differentiation of the inner automorphism leads to the adjoint operator:

\[ \text{Ad}_g : \mathfrak{g} \to \mathfrak{g}, \quad \text{Ad}_g \eta \equiv T_e I_g \cdot \eta \]

Differentiation of the adjoint operator (with respect to \(g\)) leads to the Lie bracket, sometimes denoted by \(\text{ad}\),

\[ \text{ad}_\xi \eta \equiv T_e (\text{Ad} \eta) \cdot \xi = [\xi, \eta] \]

- Transformation of observer.
- Used for body/spatial transformations.
The Exponential Map

A flow is obtained by solving for the differential equations defined by a left-invariant vector field,

\[ \dot{g} = X_\xi(g) = T_e L_g \xi = g\xi \]

This flow defines the exponential map,

\[ \exp : \mathfrak{g} \rightarrow G, \quad \xi \mapsto e^\xi \]

Keeping the time parametrization gives, \( \exp(\xi t) \).
The Exponential Map

A flow is obtained by solving for the differential equations defined by a left-invariant vector field,

\[ \dot{g} = X_\xi(g) = T_e L_g \xi = g\xi \]

This flow defines the exponential map,

\[ \exp : \mathfrak{g} \rightarrow G, \quad \xi \mapsto e^\xi \]

Keeping the time parametrization gives, \( \exp(\xi t) \).
Definition 5  Let $Q$ be a manifold and let $G$ be a Lie group. A left action of a Lie group $G$ on $M$ is a smooth mapping $\Phi : G \times Q \rightarrow Q$ such that:

1. $\Phi(e, x) = x$, $\forall x \in Q$, and

2. $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$, $\forall g, h \in G$.

The action of $g \in G$ on $q \in Q$ will typically be written as $g \cdot q$ or simply $gq$.

1. free: for all $x \in Q$, $\Phi_g(x) = x$ implies that $g = e$.

2. proper: $W \subset Q$ compact implies $\Phi^{-1}(W) \subset G \times Q$ compact.
Infinitesimal Generators

The action of $G$ on $Q$ induces a vector field on $Q$.

The Lie algebra exponential $\exp$ defines a curve on $Q$,

$$
\Phi^\xi_t(q) \equiv \exp(\xi t) \cdot q
$$

which after time differentiation,

$$
\xi_Q(q) \equiv \frac{d}{dt}\bigg|_{t=0} \exp(\xi t) \cdot q = \xi \cdot q
$$

gives the *infinitesimal generator*.
Infinitesimal Generators

The action of $G$ on $Q$ induces a vector field on $Q$.

The Lie algebra exponential $\exp$ defines a curve on $Q$,

$$\Phi_{t}^{\xi}(q) \equiv \exp(\xi t) \cdot q$$

which after time differentiation,

$$\xi_{Q}(q) \equiv \frac{d}{dt}\Big|_{t=0} \exp(\xi t) \cdot q = \xi \cdot q$$

gives the *infinitesimal generator*.
Definition 5  *Given an action of* $G$ *on* $Q$ *and* $q \in Q$, *the orbit of* $q$ *is defined by*

$$\text{Orb} \ (q) \equiv \{ \Phi_g(q) \mid g \in G \} \subset Q$$

*The tangent space at* $q$ *to the group orbit through* $q_0$ *is given by,*

$$T_q \text{Orb} \ (q_0) = \{ \xi_{Q(q)} \mid \xi \in \mathfrak{g} \}$$
**Definition 5** A principal bundle is a fiber bundle such that the model fiber is a Lie group, $G$.

For mechanical systems the base space, $M$, is sometimes called the shape space.

- Many control systems decompose this way.
- Shape $\rightarrow$ Directly controlled.
- Group $\rightarrow$ What we want to control (locomote within).
- Inherits all structures discussed.
Examples

Snakeboard
\[ \mathbb{T}^3 \times SE(2) \]

Planar Fish
\[ \mathbb{T}^2 \times SE(2) \]

Hilare Robot
\[ \mathbb{T}^2 \times SE(2) \]

Planar Amoeba
\[ \mathbb{R}^3 \times SE(2) \]