CDS 101/110 Homework #5 Solution

Problem 1 (CDS 101, CDS 110): (15 points)

From textbook page 10-3,

$$L = \frac{GZ_1}{Z_1 + Z_2}$$

Because $Z_1 = Z_2$ and $a = 0$,

$$\begin{aligned} L(s) &= \frac{G(s)}{2} \\ &= \frac{ka_1a_2}{2s(s+a_1)(s+a_2)} \end{aligned}$$

From Nyquist criteria, we know that the system is stable when there is no enclosure of the critical point -1 (i.e. $L(i\omega) > -1$ on the real line). First, find $L(i\omega)$,

$$L(i\omega) = \frac{ka_1a_2}{2i\omega(i\omega+a_1)(i\omega+a_2)} = \frac{-ika_1a_2(-i\omega+a_1)(-i\omega+a_2)}{2\omega(\omega^2+a_1^2)(\omega^2+a_2^2)} = \frac{-ika_1a_2(a_1a_2-(a_1+a_2)i\omega-\omega^2)}{2\omega(\omega^2+a_1^2)(\omega^2+a_2^2)}$$

Then, set the imaginary part of $L(i\omega) = 0$ to obtain the phase crossover frequency.

$$ka_1a_2(a_1a_2-\omega^2) = 0 \implies \omega_{pc} = \sqrt{a_1a_2}$$

Then, we can find $L(i\omega)$ at $\omega = \omega_{pc}$,

$$L(i\omega_{pc}) = \frac{-k}{2(a_1+a_2)}$$

For stability,

$$\frac{-k}{2(a_1+a_2)} > -1 \implies k < 2(a_1+a_2)$$

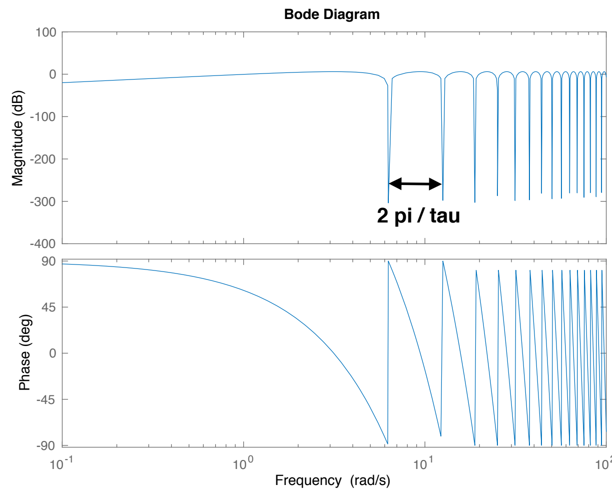
Gain margin, GM is given by

$$GM = \frac{1}{|L(i\omega_{pc})|} = \frac{2(a_1+a_2)}{k}$$

Problem 2 (CDS 101, CDS 110): (10 points)

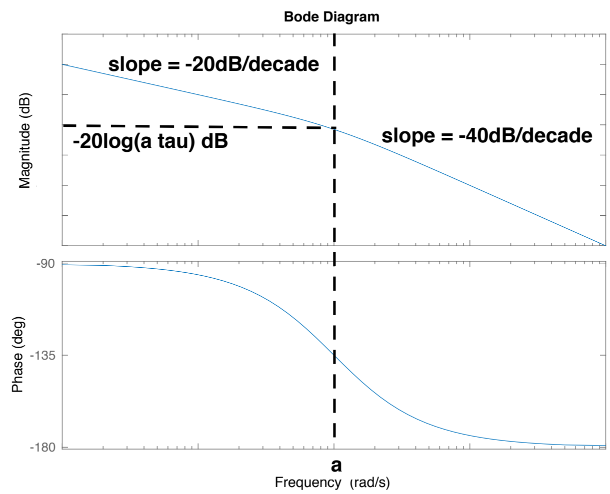
To draw a Nyquist plot, we will first find the Bode plot.

First, consider the $1-e^{-s\tau}$ term. This term is a circle centered at 1 in the complex plane. And, thus the Bode plot is given by

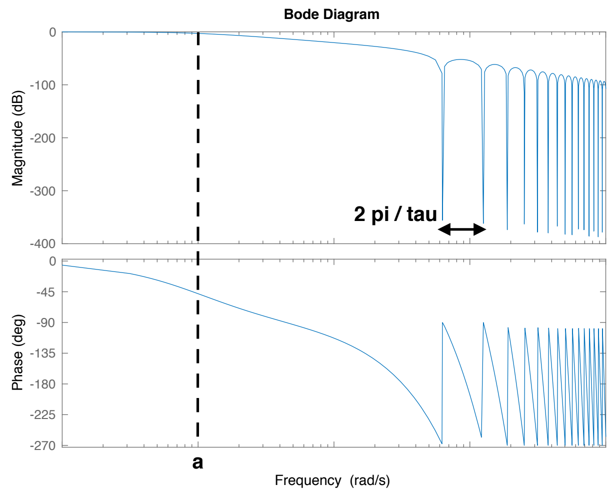


Note that the magnitude oscillates between 0 and 2 (i.e. between $-\infty$ dB and ~ 6 dB) and the phase oscialles between -90 deg and 90 deg.

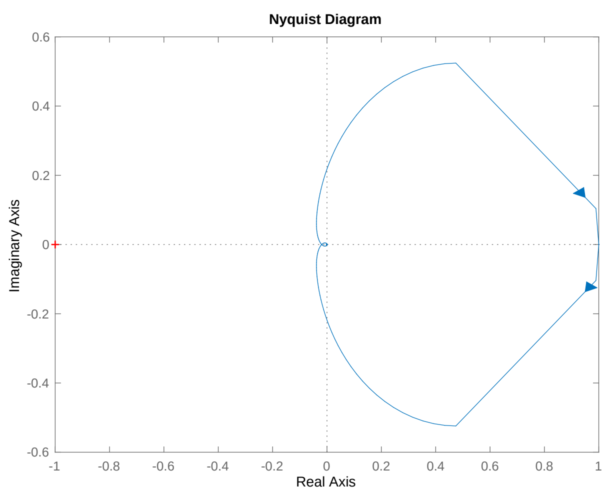
Then, consider $a/(s\tau(s+a))$ where the Bode plot is given by



Sum the two Bode plots together to obtain the Bode plot for $P(s)$



Then, the Nyquist plot of $P(s)$ is given by mapping the Bode plot into the complex plane



From the Bode plot, note that the frequency gain is the largest when $s \to 0$.

$$\lim_{s \to 0} P(s) = 1$$

Then, the maximum porportional gain $k_p \leq 1$ so that $k_p P(s)$ for all frequencies is no larger than 1 (i.e. stable). Hence, $k_p = 1$ at the boundary of stability.

You may use L'Hospital's Rule or Pade aprroximation to obtain the limit of $P(s)$. Pade approximation of the exponential is given by

$$e^{-s\tau} = \frac{1 - s\tau/2}{1 + s\tau/2}$$

Problem 3 (CDS 101, CDS 110): (30 points)

(a) Pole of closed loop response $G_{yr}(s)$ is the zero of $1 + L(s)$.

$$1 + L(s) = \frac{s(s-1) + K_2(1 + K_1 s)}{s(s-1)}$$

The zero is given by

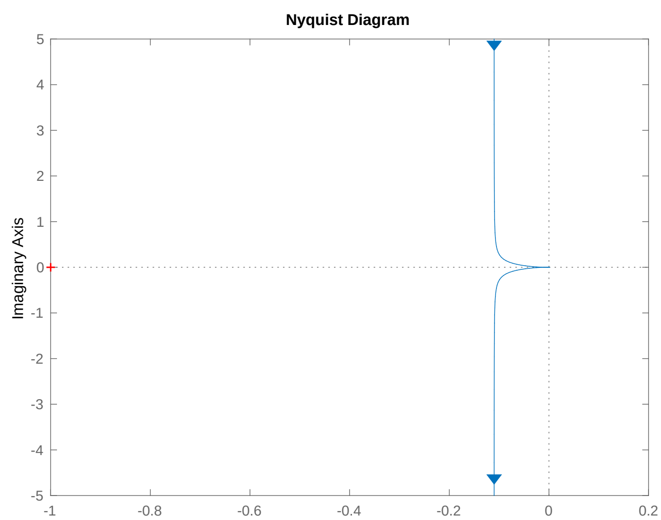
$$s(s-1) + K_2(1 + K_1 s) = 0$$

$$s = \frac{-(K_1K_2 - 1) \pm \sqrt{(K_1K_2 - 1)^2 - 4K_2}}{2}$$

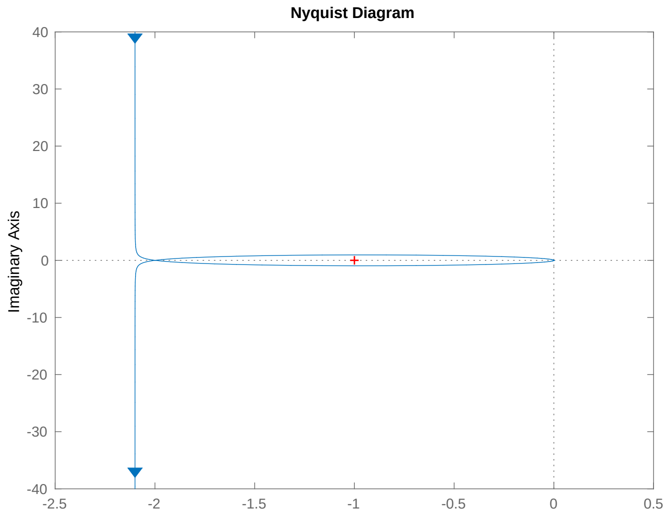
Thus, the condition for stability is $K_1K_2 - 1 > 0 \implies K_1K_2 > 1$. So, the system is stable if $K_1K_2 > 1$.

(b) Two cases:

$K_1K_2 \leq 1$



$K_1K_2 > 1$



(c) The loop transfer function $L(s)$ has one unstable pole. Therefore, we need one clockwise encirclement of the critical point -1 in the Nyquist plot of $L(i\omega)$ to obtain stability.

$$L(i\omega) = \frac{K_2(1 + K_1 i\omega)}{i\omega(i\omega - 1)} = \frac{-iK_2(1 + K_1 i\omega)(-i\omega - 1)}{\omega(\omega^2 + 1)} = \frac{-iK_2(-1 - (K_1 + 1)i\omega + K_1\omega^2)}{\omega(\omega^2 + 1)}$$

Set the imaginary part of $L(i\omega) = 0$ to obtain the phase crossover frequency.

$$K_2(-1 + K_1\omega^2) = 0 \implies \omega_{pc} = \sqrt{\frac{1}{K_1}}$$

Then, we can find $L(i\omega)$ at $\omega = \omega_{pc}$,

$$L(i\omega) = -K_1 K_2$$

For stability (i.e. one encirclement of -1), we want $-K_1K_2 < -1 \implies K_1K_2 > 1$.

Problem 4 (CDS 110): (20 points)

Note that $e^{-i\omega\tau} = \cos(\omega\tau) - i\sin(\omega\tau)$. So, we can rewrite $L(i\omega)$ as

$$L(i\omega) = \frac{k}{i\omega}(\cos(\omega\tau) - i\sin(\omega\tau)) = \frac{-ik}{\omega}(\cos(\omega\tau) - i\sin(\omega\tau))$$

Set the imaginary part of $L(i\omega) = 0$ to obtain the phase crossover frequency.

$$-\frac{k}{\omega}\cos(\omega\tau) = 0 \implies \omega_{pc} = \frac{n\pi}{2\tau}$$

Then, we can find $L(i\omega)$ at $\omega = \omega_{pc}$,

$$L(i\omega) = \mp\frac{2\tau k}{n\pi}$$

We are interested in the left most crossover. So, $n = 1$ and

$$L(i\omega) = -\frac{2\tau k}{\pi}$$

For stability,

$$-\frac{2\tau k}{\pi} < -1 \implies k > \frac{\pi}{2\tau}$$

It follows that the system will always be unstable if the time delay is too long. If the time delay is measured, the difficulty can be avoided by making the gain inversely proportional to the time delay. A reasonable choice is $k = 0.5/\tau$ which gives the stability margin $s_m = 0.63$ for all time delays τ. The response time will however be proportional to τ. Thus, we show that if we can measure the time delay, it is possible to choose a gain that gives a stability margin of $s_m \geq 0.6$ for all time delays τ.