ME/CS 133(a): Notes on Rotations

1 Spherical Kinematics

Motions of a 3-dimensional rigid body where one point of the body remains fixed are termed *spherical motions.* A *spherical displacement* is a rigid body displacement where there is a fixed point in the initial and final positions of the moving body. It can be shown that spherical displacements and spherical motions have 3 independent degrees of freedom.

Consider a 3-dimensional rigid body, $\mathcal{B}$. Let C denote the point in $\mathcal{B}$ which is fixed during subsequent spherical motions. Choose a fixed reference frame, $\mathcal{F}$ whose origin lies at point C. To understand what happens to a rigid body during a spherical motion, select an arbitrary point, P_0, in $\mathcal{B}$. Let $\vec{v}_0$ denote the vector from the origin of this fixed reference frame to P_0. I.e., $\vec{v}_0$ denotes the coordinates of P_0 as seen by an observer in frame $\mathcal{F}$. Let the body undergo a spherical displacement. After the displacement, let the new location of the point P_0 be denoted by P_1. Similarly, let $\vec{v}_1$ denote the coordinates of P_1, as seen in frame $\mathcal{F}$.

Let us assume that the transformation of the points in $\mathcal{B}$ during the spherical displacement can be represented by the action of a 3×3 matrix, A, acting on the vector of particle coordinates:

$$\vec{v}_1 = A\vec{v}_0 \ .$$

Because the body is rigid, the distance between the fixed point of the motion (which is also the origin of the fixed reference frame) and the given particle is constant. That is:

$$|\vec{v}_1| = |\vec{v}_0| \ .$$

This implies that:

$$|\vec{v}_1|^2 = |\vec{v}_0|^2 \ \Rightarrow \ \vec{v}_1^T\vec{v}_1 = \vec{v}_0^T\vec{v}_0 \ \Rightarrow \ \vec{v}_0^T A^T A\vec{v}_0 = \vec{v}_0^T\vec{v}_0.$$

Since this relationship holds for any choice of P_0, it must be true that $A^TA = I$. That is, A is an orthogonal matrix.

Recall that for matrices A and B, $det(A^T) = det(A)$ and $det(AB) = det(A)\ det(B)$. Hence, for an orthogonal matrix A, $det(A^TA) = det(A^T)det(A) = det^2(A) = det(I) = 1$. Therefore, $det(A) = \pm 1$ for any orthogonal matrix A. Those orthogonal matrices whose determinant is $+1$ are physically associated with *rotations*, while those whose determinant is -1 are associated with *reflections*. We will be primarily concerned with rotations.

2 Cayley's formula

Our goal in this section is to derive Cayley's formula, which shows that any orthogonal matrix is a specific function of a skew symmetric matrix.

From above, we have $|\vec{v}_1| = |\vec{v}_0|$, which is equivalent to:

$$|\vec{v}_1|^2 - |\vec{v}_0|^2 = 0 = \vec{v}_1 \cdot \vec{v}_1 - \vec{v}_0 \cdot \vec{v}_0 = (\vec{v}_1 - \vec{v}_0) \cdot (\vec{v}_1 + \vec{v}_0)$$

Hence, the vectors $(\vec{v}_1 - \vec{v}_0)$ and $(\vec{v}_1 + \vec{v}_0)$ are orthogonal to each other. Let $\vec{d}^+ = \vec{v}_1 + \vec{v}_0 = (A + I)\vec{v}_0$ and $\vec{d}^- = \vec{v}_1 - \vec{v}_0 = (A - I)\vec{v}_0$. The vectors $\vec{d}^+$ and $\vec{d}^-$ are the diagonals on a rhombus whose sides are parallel to $\vec{v}_0$ and $\vec{v}_1$. Solving for $\vec{v}_0$ in terms of $\vec{d}^+$, it is easy to see that

$$\vec{d}^- = (A - I)\vec{v}_0 = (A - I)(A + I)^{-1}\vec{d}^+.$$

If we introduce the notation

$$B = (A - I)(A + I)^{-1} \tag{1}$$

where see that $\vec{d}^- = B\vec{d}^+$. Note B is a matrix that operates on a vector, $\vec{d}^+$, to produce an orthogonal vector, $\vec{d}^-$. Note that this derivation was independent of any particular choice of the point P_0. Thus, the matrix B must generally have the property that

$$\vec{y}^T B \vec{y} = 0 = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{31} & b_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \sum_{i=1}^{3} b_{ii} y_i^2 + \sum_{i=1}^{3} \sum_{j=i}^{3} (b_{ij} + b_{ji}) y_i y_j \tag{2}$$

for any 3×1 vector $\vec{y}$. For B to satisfy Equation (2), it must generally be a skew symmetric matrix:

$$\begin{bmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{bmatrix} \tag{3}$$

which contains only 3 independent entries. We can solve Equation (1) for A to obtain *Cayley's formula*:

$$A = (I - B)^{-1}(I + B). \tag{4}$$

Due to the fact that A is an orthogonal matrix (which implies that $A^T = A^{-1}$) and the skew symmetry of B, we can derive the following equivalent formula

$$A = (I + B)(I - B)^{-1}.$$

That is, the matrices $(I + B)$ and $(I - B)^{-1}$ commute.

3 Eigenvalues/Eigenvectors of a Rotational Matrix

Let A by a 3×3 orthogonal matrix, where A has the symbolic form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} . \tag{5}$$

The eigenvalues of A can be determined from the roots of the characteristic polynomial

$$det(A - \lambda I) = 0 \tag{6}$$

A brute force calculation results in

$$det(A-\lambda I) = -\lambda^3 + \lambda^2(a_{11}+a_{22}+a_{33}) - \lambda[cof(a_{11})+cof(a_{22})+cof(a_{33})] + det(A) = 0. \quad (7)$$

where $cof(a_{ii})$ is the cofactor the a_{ii}, i.e., the determinant of the 2×2 matrix formed from elimination the i^{th} row and i^{th} column from A. Note that for $A \in SO(3)$, $det(A) = 1$.

One can also show that $cof(a_{ii}) = a_{ii}$ for matrices in $SO(3)$. To see this, note that the columns of A are unit vectors, which can be denoted as:

$$\mathbf{x} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \mathbf{y} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \mathbf{z} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

These columns can be interpreted as the unit vectors of an orthogonal right handed coordinate system. Consequently,

$$\begin{aligned} \mathbf{x} &= \mathbf{y} \times \mathbf{z} \\ \mathbf{y} &= \mathbf{z} \times \mathbf{x} \\ \mathbf{z} &= \mathbf{x} \times \mathbf{y} \end{aligned}$$

Performing the cross product and equating sides for $\mathbf{x} = \mathbf{y} \times \mathbf{z}$, we get the relation:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} \\ a_{13}a_{32} - a_{12}a_{33} \\ a_{12}a_{23} - a_{13}a_{22} \end{bmatrix} = \begin{bmatrix} \text{cofactor}(a_{11}) \\ \text{cofactor}(a_{21}) \\ \text{cofactor}(a_{31}) \end{bmatrix}$$

Similar relationships can be derived for the other columns to show that $a_{ij} = \text{cofactor}(a_{ij})$ for all elements of a special orthogonal matrix.

Hence, Equation (7) reduces to

$$det(A - \lambda I) = -(\lambda - 1)[\lambda^2 - \lambda(a_{11} + a_{22} + a_{33} - 1) + 1] = 0\ . \quad (8)$$

Letting

$$\cos\phi = \frac{a_{11} + a_{22} + a_{33} - 1}{2}, \quad (9)$$

Equation (8) can be rewritten as

$$det(A - \lambda I) = -(\lambda - 1)[\lambda^2 - 2\cos\phi\lambda + 1] = 0\ . \quad (10)$$

The eigenvalues of $A \in SO(3)$ are therefore:

$$\begin{aligned} \lambda_1 &= 1 & (11) \\ \lambda_{2,3} &= \cos\phi \pm j\sin\phi = e^{\pm j\phi} & (12) \end{aligned}$$

Note that λ_2 and λ_3 are complex conjugates.

3.1 Physical Interpretation of the Eigenvalues/Eigenvectors

Let $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ denote the eigenvectors associated with eigenvalues λ_1, λ_2, and λ_3. Recall from the definition of an eigenvalue/eigenvector that

$$A\vec{e}_j = \lambda_j \vec{e}_j.$$

The eigenvalues of rotation matrices can be physically interpreted as follows. The eigenvector $\vec{e}_1$ is termed the "*axis of rotation*," since $A\vec{e}_1 = \lambda_1\vec{e}_1 = \vec{e}_1$. That is, $\vec{e}_1$ is an *invariant* of A. If A represents a rotational displacement of a body $\mathcal{B}$, then the set of points in $\mathcal{B}$ that lie along the line passing through the fixed point of the rotation and in the direction of $\vec{e}_1$ remain fixed by the displacement.

The eigenvectors $\vec{e}_2$ and $\vec{e}_3$ are generally complex, and will be complex conjugates (since λ_2 and λ_3 are complex conjugates). To interpret these complex eigenvalues/eigenvectors, construct the real vectors:

$$\vec{c}_2 = \frac{1}{2}(\vec{e}_2 + \vec{e}_3) \quad (13)$$

$$\vec{c}_3 = \frac{i}{2}(\vec{e}_2 - \vec{e}_3) \ . \quad (14)$$

Note that $\vec{c}_2$ is orthogonal to $\vec{c}_3$, and both $\vec{c}_2$ and $\vec{c}_3$ are orthogonal to $\vec{e}_1$. Thus $\vec{c}_2$ and $\vec{c}_3$ span a plane that is orthogonal to the axis of rotation.

Let's compute the action of A on $\vec{c}_2$ and $\vec{c}_3$:

$$\begin{aligned} A\vec{c}_2 &= \frac{1}{2}\ A(\vec{e}_2 + \vec{e}_3) = \frac{1}{2}(\lambda_2\vec{e}_2 + \lambda_3\vec{e}_3) = \frac{1}{2}(\cos\phi + j\sin\phi)\vec{e}_2 + \frac{1}{2}(\cos\phi - j\sin\phi)\vec{e}_3 \\ &= \vec{c}_2\cos\phi \ + \ \vec{c}_3\sin\phi \quad (15) \\ A\vec{c}_3 &= \frac{i}{2}\ A(\vec{e}_2 - \vec{e}_3) = -\vec{c}_2\sin\phi + \vec{c}_3\cos\phi \quad (16) \end{aligned}$$

Thus, A acts to rotate vectors by angle ϕ about the axis $\vec{e}_1$. We can summarize these findings in *Euler's theorem.*

Theorem 1 *(***Euler's Theorem***). Every rigid body rotational displacement is equivalent to a "simple" rotation about a fixed axis.*

Because there are a great many theorems associated with the name of Euler, this theorem is sometimes termed *Euler's Displacement Theorem.*

4 Formulas for the Rotation Matrix

So far we have developed Cayley's formula, which shows that a 3×3 orthogonal matrix can be expressed as a function of a 3×3 skew symmetric matrix, which has only 3 independent parameters. We have also characterized rotations in terms of Euler's theorem, which suggests a parametrization of rotations in terms of a fixed axis and an angle. We now wish to derive a formula for a rotation matrix in terms of this axis and angle. Our goal will be to find expressions for the entries of the skew symmetric matrix B in terms of the rotation axis and rotation angle.

By relabeling the entries of B in Equation (3), we ca assume that B has the form:

$$\begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}. \tag{17}$$

Note that if $\vec{b} = [b_1 \ b_2 \ b_3]^T$, then $B\vec{v} = \vec{b} \times \vec{v}$ for any 3×1 vector $\vec{v}$.

Recall that if $\vec{e}_j$ is an eigenvector of A with eigenvalue λ_j, then $(A - \lambda_j I)\vec{e}_j = 0$. Let's apply this relationship to the axis of rotation, whose associated eigenvalue has unit value. Multiply both sides of Cayley's formula:

$$(A - I)\vec{e}_1 = [(I - B)^{-1}(I + B) - I]\vec{e}_1 = 0, \tag{18}$$

by the quantity $(I - B)$, and then simplify to obtain

$$2B\vec{e}_1 = 0.$$

This equation implies that $\vec{b}$ is parallel (or antiparallel) to $\vec{e}_1$: $\vec{b} = \pm c\vec{e}_1$, where c is an as yet undetermined constant. The constant c is found as follows.

Recall our derivation of Cayley's formula from Section 2. Let D denote a plane that is perpendicular to the axis of rotation, and passing through the fixed point of the rotation. From the discussion above, it should be clear that D is also perpendicular to $\vec{b}$. Let $\vec{v}_1^*$ and $\vec{v}_0^*$ denote the projection of vectors $\vec{v}_1$ and $\vec{v}_0$ onto D. Recall from above that $\vec{v}_1 - \vec{v}_0 = \vec{b} \times (\vec{v}_1 - \vec{v}_0)$. This relationship will continue to hold for the projected vectors:

$$\vec{v}_1^* - \vec{v}_0^* = \vec{b} \times (\vec{v}_1^* - \vec{v}_0^*). \tag{19}$$

Let ϕ denote the angle between $\vec{v}_0^*$ and $\vec{v}_1^*$. By observation,

$$\tan(\frac{\phi}{2}) = \frac{|\vec{v}_1^* - \vec{v}_0^*|/2}{|\vec{v}_1^* + \vec{v}_0^*|/2} = \frac{|\vec{v}_1^* - \vec{v}_0^*|}{|\vec{v}_1^* + \vec{v}_0^*|}$$

If we take the norm of both sides of Equation (19), and recall that $\vec{b}$ is perpendicular to $(\vec{v}_1^* + \vec{v}_0^*)$, we get

$$|\vec{b}| = \frac{|\vec{v}_1^* - \vec{v}_0^*|}{|\vec{v}_1^* + \vec{v}_0^*|} = \tan(\frac{\phi}{2})$$

Hence, $\vec{b} = \tan\left(\frac{\phi}{2}\right)\vec{e}_1$. Hereafter, let $\vec{\omega}$ denote a unit vector collinear with the axis of rotation. Then

$$\vec{b} = \tan(\frac{\phi}{2})\vec{\omega}.$$

Let $\hat{\omega}$ be the 3×3 skew symmetric matrix such that $\hat{\omega}\vec{v} = \vec{\omega} \times \vec{v}$ for any vector $\vec{v}$:

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ -\omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{20}$$

Hence, $B = \tan(\frac{\phi}{2})\ \hat{\omega}$. Substituting this result into Equation (4) gives us an explicit formula for a rotation matrix in terms of the axis of rotation, $\vec{\omega}$ and angle of rotation, ϕ.

$$A = (I - B)^{-1}(I + B) = (I - \tan(\frac{\phi}{2})\hat{\omega})^{-1}(I + \tan(\frac{\phi}{2})\hat{\omega})$$

Some algebraic rearrangement of this formula yields *Rodriguez' Formula* for a rotation:

$$A = I + \sin\phi\ \hat{\omega} + (1 - \cos\phi)\ \hat{\omega}^2.$$

Thus, Rodriguez' Formula gives us the explicit parametrization of a rotation matrix in terms of the Axis of Rotation and the Angle of Rotation (Euler's displacement theorem).