CDS 101/110 Homework #2 Solution

Problem 1 (CDS 101, CDS 110): (20 points for CDS 101, 25 Points for CDS 110)

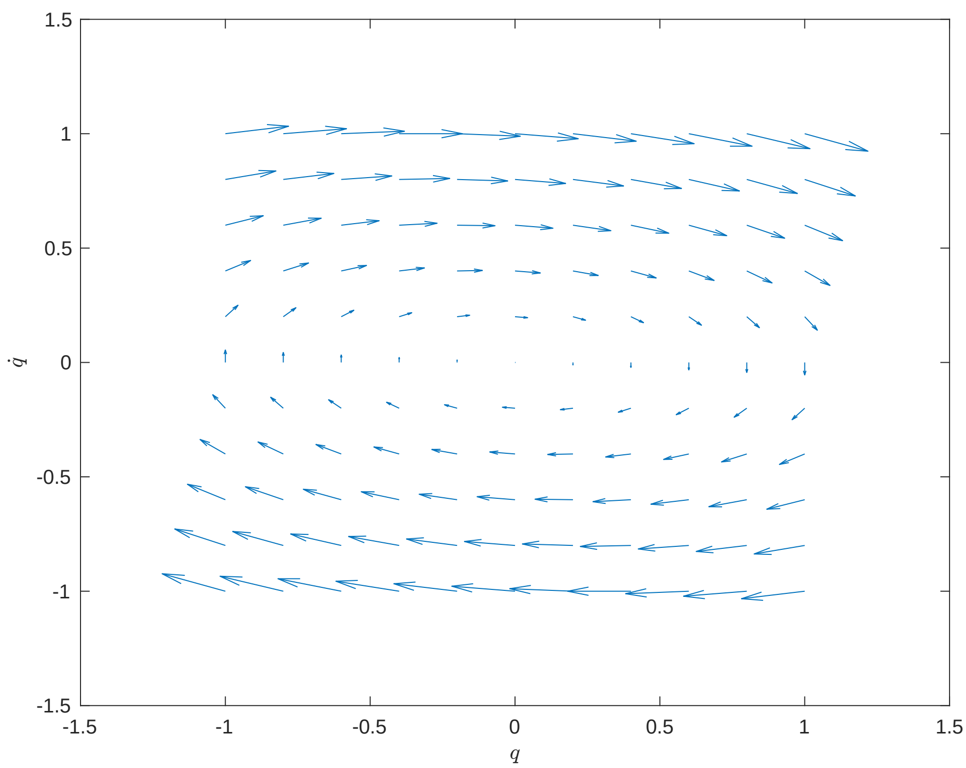
(a) At equilibrium, $\ddot{q} = \dot{q} = 0$. So, $m\ddot{q} = -k(q + aq^3) - b\dot{q}$ becomes $0 = -k(q + aq^3)$. Solve for q to get $q = 0$. Ignore the imaginary q because this is a physical system.

Therefore, the equilibirum point is $(q, \dot{q}) = (0, 0)$. Note that $q = 0$ alone is not enough.

(b) Code:

```
m = 1000;
k = 250;
a = 0.02;
b = 100;
[x1,x2]=meshgrid(-1:0.2:1, -1:0.2:1);
x1dot = x2;
x2dot = (-(k/m)*(x1+a*x1^3))-b*x2/m;
quiver(x1,x2,x1dot,x2dot);
xlabel('$q$','Interpreter','latex')
ylabel('$\dot{q}$','Interpreter','latex')
```

Plot:



(c) Let $x = (x_1, x_2)$ where $x_1 = q$ and $x_2 = \dot{q}$. Rewrite $m\ddot{q} = -k(q + aq^3) - b\dot{q}$ as

$$\frac{dx}{dt} \triangleq F = \begin{bmatrix} x_2 \\ -\frac{k}{m}(x_1 + ax_1^3) - \frac{b}{m}x_2 \end{bmatrix}$$

Take the derivative:

$$\frac{dF}{dx} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m}(1 + 3ax_1^2) & -\frac{b}{m} \end{bmatrix}$$

At equilibrium,

$$\frac{dF}{dx} \triangleq A \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}$$

Thus, the linearized system is

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} x$$

(d) The eigenvalues of A are $-0.05 \pm 0.4975i$. The real parts are all less than zero. Therefore, the system is assymtotically stable.

(e) Solution to linear system can be rewrite as an exponential function. Since the real parts of all eigenvalues of A are less than zero, the system is decaying exponentially. Therefore, the system is exponentially stable.

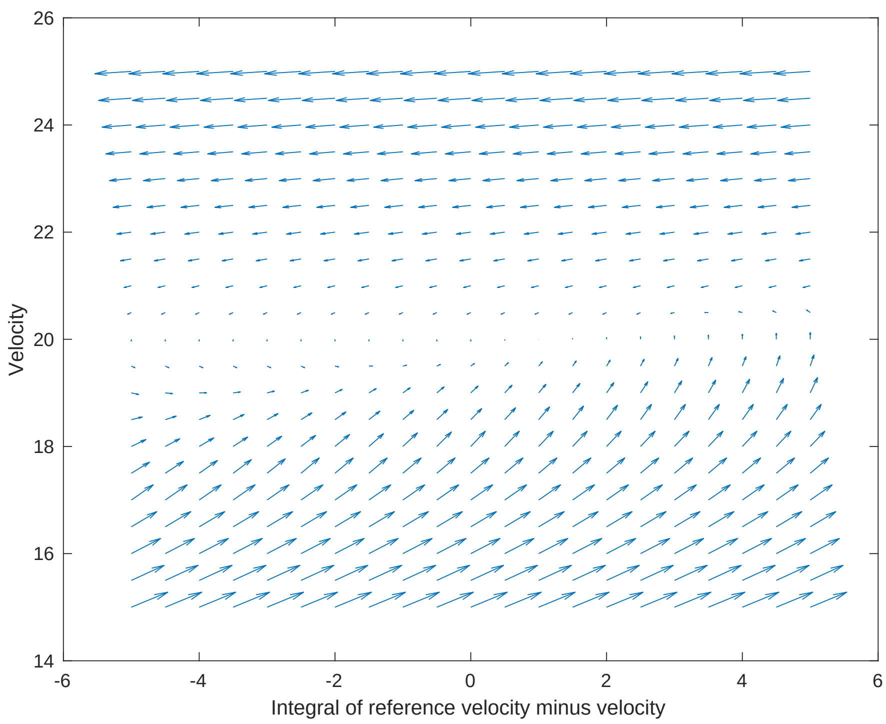
Problem 2 (CDS 101, CDS 110): (15 points)

Code:

```
% x1 represents the reference velocity minus velocity
% x2 represents the velocity

[x1,x2]=meshgrid(-5:0.5:5, 15:0.5:25);

% Set the reference speed and the current error
Vr = 20;
Verr = Vr - x2;

% Gain values
Kp = .5; Ki = .1;

% Set the throttle
u = Kp*Verr + Ki*x1;

% Saturate the input of the throttle
u = min(u, 1);  u = max(0, u);

% Parameters for defining the system dynamics (agree with simulink model)
m = 1000;               % mass of the vehcile
alpha = 16; % gear ratios
Tm = 190; % engine torque constant, Nm
wm = 420; % peak torque rate, rad/sec
```

```
beta = 0.4; % torque coefficient
Cr = 0.01; % coefficient of rolling friction
rho = 1.3; % density of air, kg/m?3
Cd = 0.32; % drag coefficient
A = 2.4; % car area, m?2
g = 9.8; % gravitational constant
gear = 3;               % choose the desired gear
theta = 0;              % the angle of the surface you?re driving on

% Determine the driving force
omega = alpha*x2;
torque = u .* Tm .* ( 1 - beta * (omega/wm - 1).^2 );
F = alpha * torque;

% Determine the opposing forces
Fr = m * g * Cr;                    % Rolling friction
Fa = 0.5 * rho * Cd * A * (x2).^2;  % Aerodynamic drag
Fg = m * g * sin(theta);            % Road slope force
Fd = Fr + Fa + Fg;                  % total deceleration

% Determine the right side of the differential equation
dx1 = Verr;
dx2 = (F - Fd)/m;


% Plot
quiver(x1,x2,dx1,dx2)
ylabel('Velocity')
xlabel('Integral of reference velocity minus velocity')
```

Plot:



Problem 3 (CDS 110): (15 points)

(a) Clearly $V_1(x)$ is positive definite. To examine $\dot{V}_1(x)$, taking the derivative of V_1 and substituting the dynamics, we have

$$\frac{dV_1(x)}{dt} = -ax_1^2 - bx_1x_2 - cx_2$$

To check whether this is negative definite, we complete the square by writing

$$\frac{dV_1(x)}{dt} = -a(x_1 + \frac{b}{2a}x_2)^2 - (c - \frac{b^2}{4a})x_2^2$$

Since $a > 0$, clearly V_1 is negative semidefinite if $c - \frac{b^2}{4a} \geq 0$.

(b) Consider $V_2(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + \frac{b}{c-a}x_1)^2$. It is easy to show that $V_2(x)$ is positive definite since $V_2(x) \geq 0$ for all x and $V_2(x) = 0$ implies that $x_1 = 0$ and $x_2 + \frac{b}{c-a}x_1 = x_2 = 0$. Note that here we require that $c \neq a$.

We now check the time derivative of V_2:

$$\begin{aligned}
\frac{dV_2(x)}{dt} &= x_1\dot{x}_1 + (x_2 - \frac{b}{c-a}x_1)(\dot{x}_2 + \frac{b}{c-a}\dot{x}_1) \\
&= -ax_1^2 + (x_2 + \frac{b}{c-a}x_1)(-bx_1 - cx_2 - \frac{ab}{c-a}x_1) \\
&= -ax_1^2 + (x_2 + \frac{b}{c-a}x_1)(-cx_2 - \frac{bc}{c-a}x_1) \\
&= -ax_1^2 + -c(x_2 + \frac{b}{c-a}x_1)^2
\end{aligned}$$

We see that $V_2(x)$ is negative definite. Hence, we show that $V_2(x)$ is a Lyapunov function as long as $c \neq a$.

Problem 4 (CDS 110): (15 points)

Elimination of u gives the following differential equation for the closed loop system

$$\frac{dx}{dt} = \begin{bmatrix} k & 1 \\ -4k & -3 \end{bmatrix} x$$

This equation has the characteristic polynomial

$$\lambda(s) = (s - k)(s + 3) + 4k = s^2 + (3 - k)s + k$$

Solvig for s by setting $\lambda(s) = 0$, we get $s = 0.5(k - 3 \pm \sqrt{9 + k^2 - 10k})$.

Code:

```
k = 0:0.01:10;

root_1 = 0.5*(k-3 + sqrt(9+k.^2-10*k));
root_2 = 0.5*(k-3 - sqrt(9+k.^2-10*k));

plot(real(root_1),imag(root_1),real(root_2),imag(root_2))
xlabel('Real')
ylabel('Imaginary')
```

Plot: