CDS 101/110: Lecture 2.1
Feedback Characteristics (continued)
Intro to Modeling

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Goals:

- Conclude hands-on investigation of Feedback Characteristics
- Review basic concepts on systems modeling (Chapter 3)
- Define a "model" and use it to answer questions about a system
- Introduce concepts of state, dynamics, inputs, and outputs

Reading:

- Åström and Murray, *Feedback Systems (2nd ed. Beta*)
  - Sections 2.1-2.4 (feedback characteristics)
  - Sections 3.1-3.2, (review of modeling for control)
  - *Optional: Sections.* 3.3-3.4 (more advanced modeling topics)

Some Characteristics of Feedback

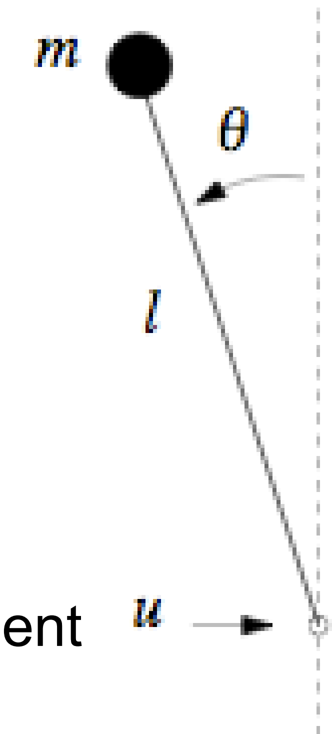
To get a “first look” at some of the issues in feedback control, last time we looked at a simple *inverted pendulum* example

- Dynamical Equation:

$$\ddot{\theta} + \frac{\varepsilon}{ml^2}\dot{\theta} - \frac{g}{l}\sin\theta = \frac{1}{ml}u \quad \rightarrow \quad \ddot{\theta} + \alpha\dot{\theta} - \beta\theta = \gamma u$$

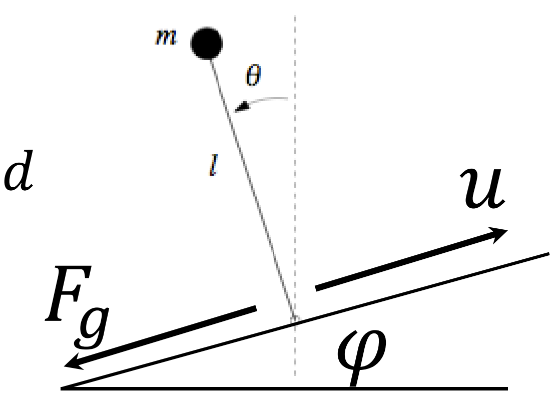
  - Where $\alpha = \frac{\varepsilon}{ml^2}, \beta = \frac{g}{l}, \gamma = \frac{1}{ml}$

- Our feedback analysis from last time:
  - Proportional feedback stabilizes, but *slow* response
  - Proportional + Derivative allows arbitrary pole placement

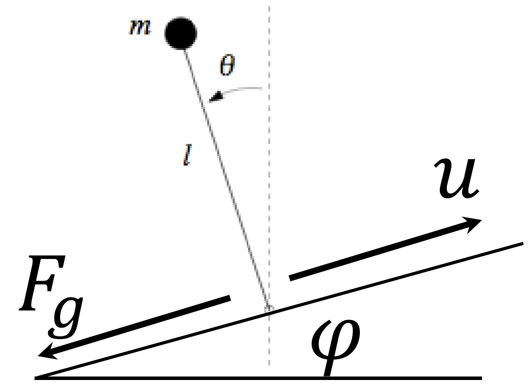


What can go wrong? *Unmodeled dynamics*

$$\ddot{\theta} + \alpha\dot{\theta} - \beta\theta = \gamma u - \frac{g}{l^2} sin\varphi \equiv \gamma u + d$$



Unmoded Dynamics



What happens with proportional feedback?

$$\ddot{\theta} + \alpha\dot{\theta} + \beta'\theta = d$$

- At equilibrium, not possible for $\left(\theta, \dot{\theta}, \ddot{\theta}\right) = 0$
- There is a solution $\left(\dot{\theta}, \ddot{\theta}\right) = 0$, and $\theta_{eq} = \frac{d}{\beta'} = -\frac{mgsin\varphi}{mgl - k_p l}$
  - Note: $-mgsin\varphi + \gamma u \neq 0 \quad \rightarrow$ *base is always moving*

Solution: *Feedforward*

- $u = \frac{mgsin\varphi}{l} - k_p\theta$
- *not robust, since m, g, l, φ must be known*

Solutions: *Integral Feedback*

- $u = -k_p\theta - k_i \int_0^t \left(\theta(\tau) - \theta_{ref}\right) d\tau = -k_p\theta - k_i \int_0^t \theta(\tau) d\tau$
- Key Idea: integrator will estimate the required bias
- System can stabilize to $\left(\theta, \dot{\theta}, \ddot{\theta}\right) = 0$, even though $\int_0^t \theta(\tau) d\tau \neq 0$
- Value of integral will converge to $mgsin\varphi / lk_i$ without knowing m, g, l, φ

Feedback Characteristics: *Take-away*

Feedback is used for

- ***Regulation:*** maintain an output variable at a fixed value
- ***Disturbance Rejection:***
- ***Trajectory*** *(Command)* ***Tracking****: (see FBS-2e, Section 2.3)*

Feedback characteristics

- Feedback one or more dynamical states
- Can set behavior of feedback controlled system
    - possibly set poles of closed loop system
- Can overcome unmodeled dynamics or imprecisely known system parameters

Model-Based Analysis of Feedback Systems

Analysis and design based on *models*

- A model can *predict* how a system will behave
- Feedback can give counter-intuitive behavior; models help sort out what is going on
- For control design, models don't have to be exact: *feedback* provides robustness

Control-oriented models: *inputs* and *outputs*

The model you use depends on the questions you want to answer

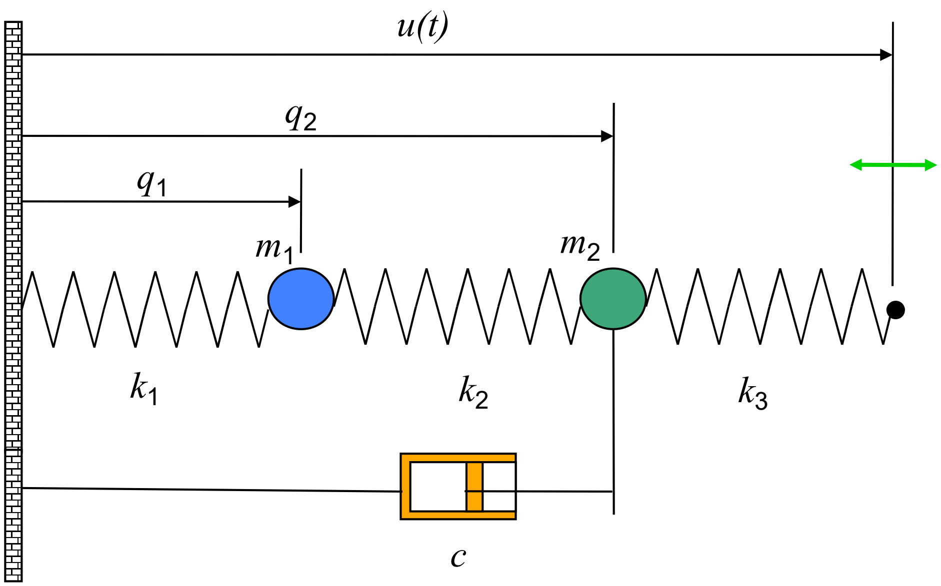
- A single system may have many models
- Time and spatial scale must be chosen to suit the questions you want to answer
- Formulate questions *before* building a model

Weather Forecasting

- **Question 1: how much will it rain tomorrow?**
- **Question 2: will it rain in the next 5-10 days?**
- **Question 3: will we have a drought next summer?**

Different questions ® different models

Example #1: Spring Mass System



Applications

- Flexible structures (many apps)
- Suspension systems (eg, “Bob”)
- Molecular and quantum dynamics

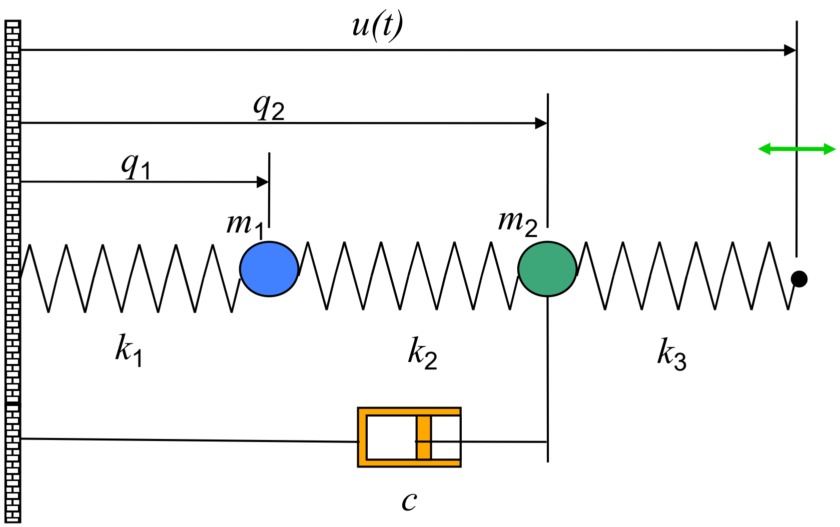
Questions we want to answer

- How much do masses move as a function of the forcing frequency?
- What happens if I change the values of the masses?
- Will Bob fly into the air if I take that speed bump at 25 mph?

Modeling assumptions

- Mass, spring, and damper constants are fixed and known
- Springs satisfy Hooke’s law
- Damper is (linear) viscous force, proportional to velocity

Modeling a Spring Mass System



Model: rigid body physics

- Sum of forces = mass * acceleration
- Hooke's law: $F = k(x - x_{\text{rest}})$
- Viscous friction: $F = c\,v$

$$m_1\ddot{q}_1 = k_2(q_2 - q_1) - k_1 q_1$$
$$m_2\ddot{q}_2 = k_3(u - q_2) - k_2(q_2 - q_1) - c\dot{q}_2$$

Convert to state space form

- Construct a *vector* of variables that specify the system's evolution
- Write dynamics as a *system* of first order differential equations:

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \frac{k_2}{m}(q_2 - q_1) - \frac{k_1}{m} q_1 \\ \frac{k_3}{m}(u - q_2) - \frac{k_2}{m}(q_2 - q_1) - \frac{c}{m}\dot{q} \end{bmatrix}$$

$$y = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

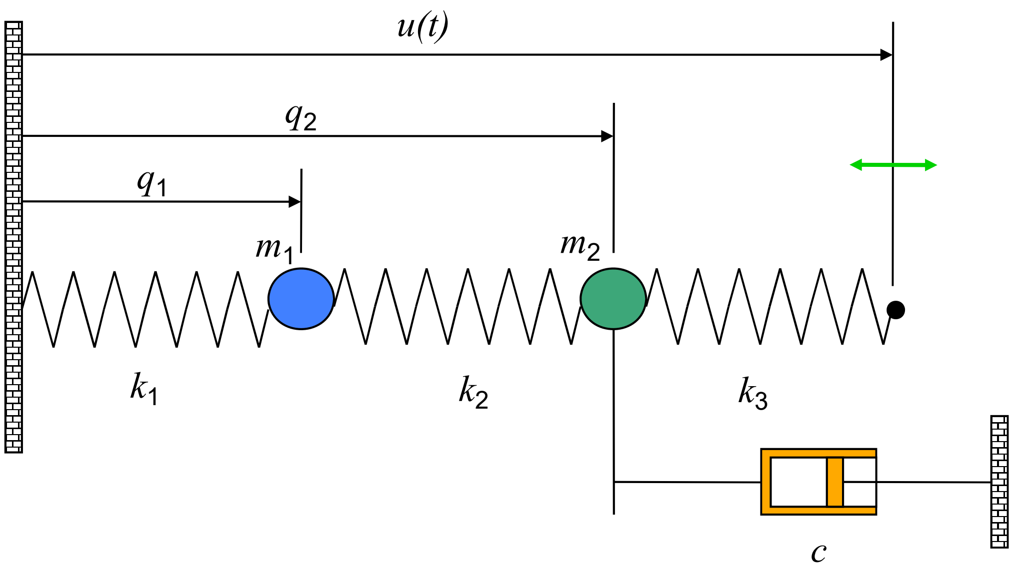
"State space form"

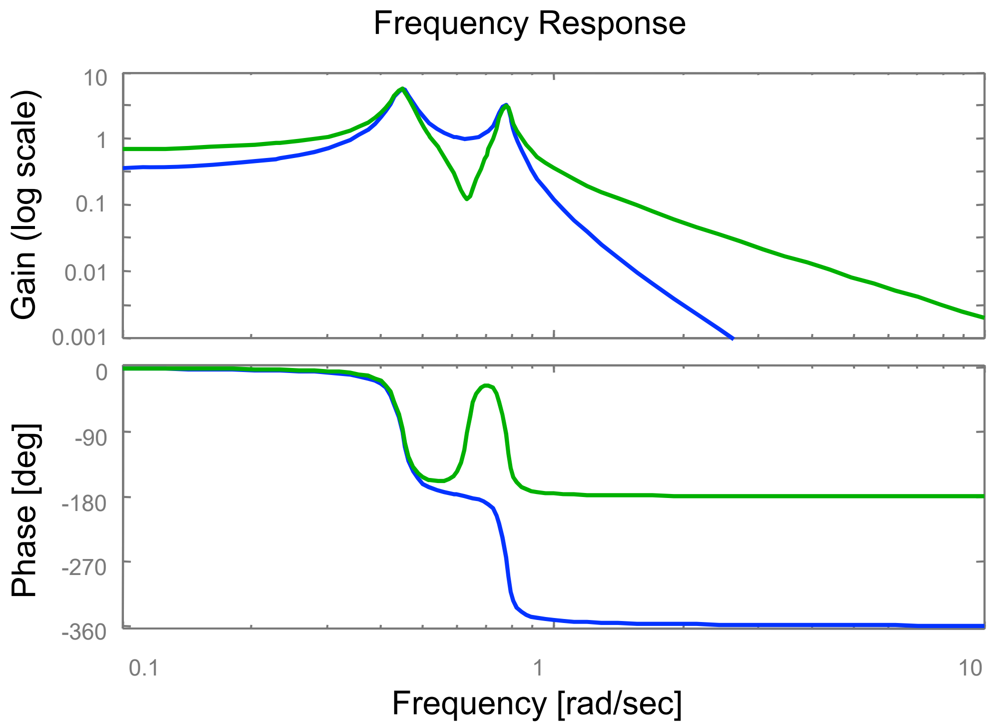
$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1 + k_2}{m} & \frac{k_2}{m} & 0 & 0 \\ -\frac{k_2}{m} & -\frac{k_2 + k_3}{m} & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_3}{m} \end{bmatrix} u$$

$$y = [1 \quad 1 \quad 0 \quad 0]x = Cx$$

$$\dot{x} = Ax + Bu$$

Simulation of a Mass Spring System





Steady state frequency response

- Force the system with a sinusoid
- Plot the “steady state” response, after transients have died out
- Plot relative magnitude and phase of output versus input (more later)

Matlab simulation (see handout)

```
function dydt = f(t, y, ...)
u = 0.00315*cos(omega*t);
dydt = [
   y(3);
  y(4);
   -(k1+k2)/m1*y(1) + k2/m1*y(2);
   k2/m2*y(1) - (k2+k3)/m2*y(2)
        - c/m2*y(4) + k3/m2*u ];

[t,y] = ode45(dydt,tspan,y0,[],
k1, k2, k3, m1, m2, c, omega);
```

More General Forms of Differential Equations

State space form

$$\frac{dx}{dt} = f(x,u)$$
$$y = h(x,u)$$

General form

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$

Linear system

$$x \in \mathbb{R}^n, u \in \mathbb{R}^p$$
$$y \in \mathbb{R}^q$$

- x = state; *n*th order
- u = input; will usually set p = 1
- y = output; will usually set q = 1

Higher order, linear ODE

$$\frac{d^n q}{dt^n} + a_1 \frac{d^{n-1} q}{dt^{n-1}} + \cdots + a_n q = u$$
$$y = b_1 \frac{d^{n-1} q}{dt^{n-1}} + \cdots + b_{n-1}\dot{q} + b_n q$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} d^{n-1}q/dt^{n-1} \\ d^{n-2}q/dt^{n-2} \\ \vdots \\ dq/dt \\ q \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} x$$

Modeling Terminology

***State* captures effects of the past**

- independent physical quantities that determines future evolution (absent external excitation)

***Inputs* describe external excitation**

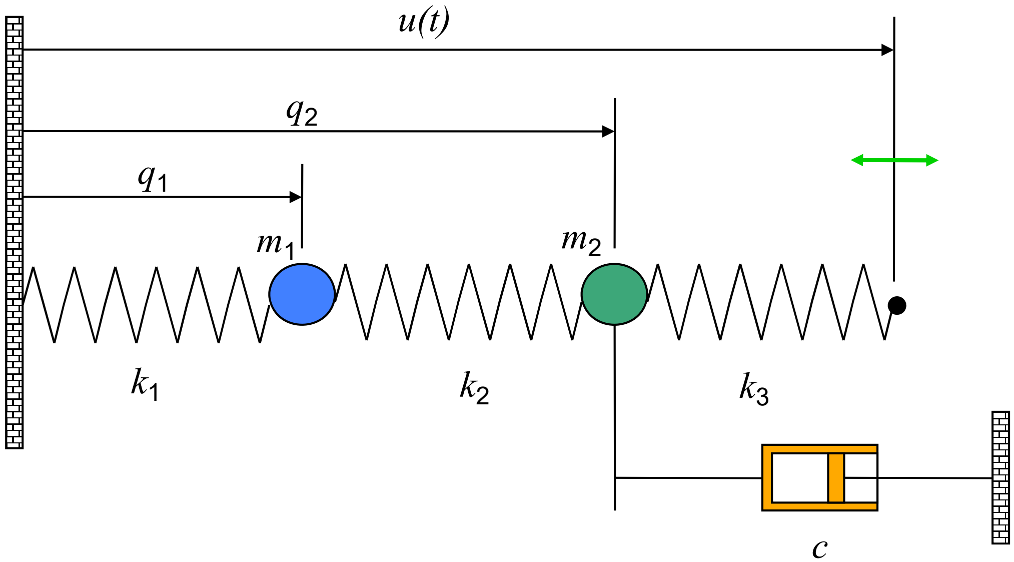
- Inputs are *extrinsic* to the system (externally specified)

***Dynamics* describes state evolution**

- update rule for system state
- function of current state and any external inputs

***Outputs* describe measured quantities**

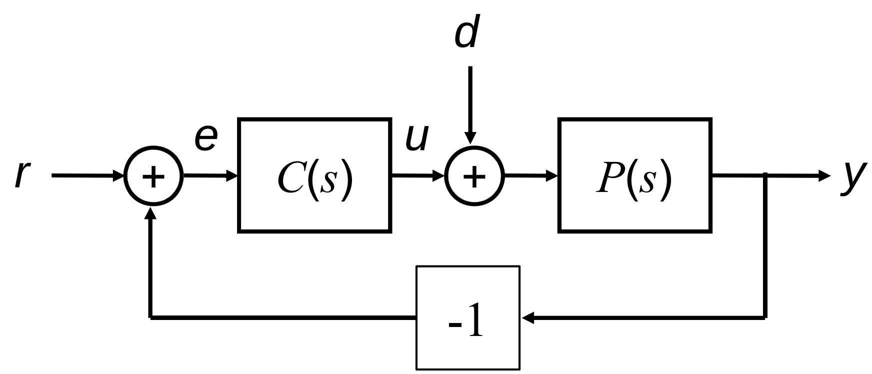
- Outputs are function of state and inputs ⇒ not independent variables
- Outputs are often *subset* or *mixture* of state



Example: spring mass system

- State: position and velocities of each mass: $q_1, q_2, \dot{q}_1, \dot{q}_2$
- Input: position of spring at right end of chain: $u(t)$
- Dynamics: basic mechanics
- Output: measured positions of the masses: q_1, q_2

Modeling Properties



Choice of state is not unique

- There may be *many* choices of variables that can act as the state
- Trivial example: different choices of units (scaling factor)
- Less trivial example: sums and differences of the mass positions

Choice of inputs, outputs depends on point of view

- Inputs: what factors are *external* to the model that you are building
  - Inputs in one model might be outputs of another model (eg, the output of a cruise controller provides the input to the vehicle model)
- Outputs: what physical variables (often states) can you *measure*
  - Choice of outputs depends on what you can sense and what parts of the component model interact with other component models

Can also have different *types* of models

- Ordinary differential equations for rigid body mechanics
- Finite state machines for manufacturing, Internet, information flow
- Partial differential equations for fluid flow, solid mechanics, etc

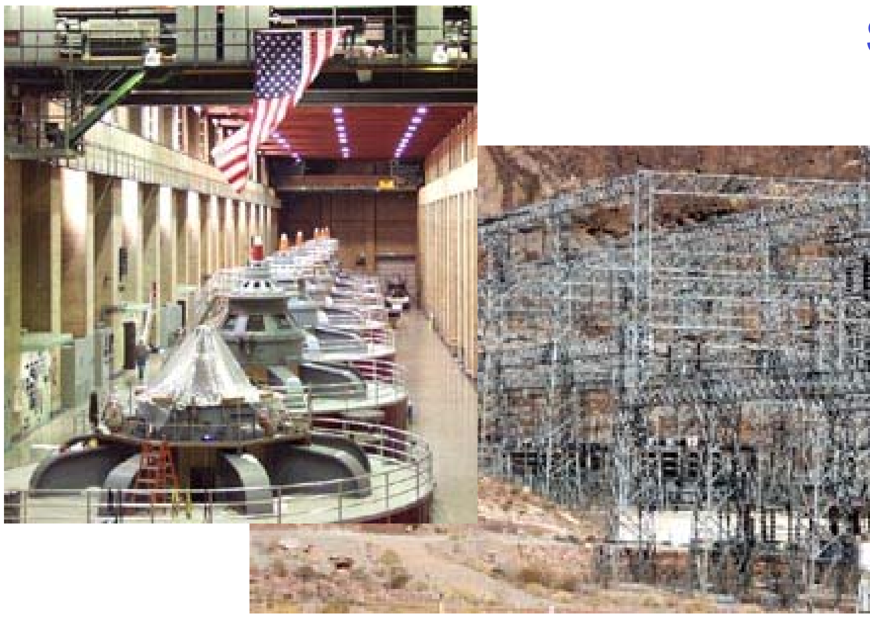
Differential Equations

Differential equations model continuous evolution of state variables

- Describe the rate of change of the state variables
- Both state and time are continuous variables

$$\frac{dx}{dt} = f(x,u)$$
$$y = h(x)$$

Example: electrical power grid



Swing equations

$$\ddot{\delta}_1 + D_1\dot{\delta}_1 = \omega_0(P_1 - B\sin(\delta_1 - \delta_2) + G\cos(\delta_1 - \delta_2)$$
$$\ddot{\delta}_2 + D_2\dot{\delta}_2 = \omega_0(P_2 + B\sin(\delta_1 - \delta_2) + G\cos(\delta_1 - \delta_2)$$

- Describe how generator rotor angles (™$_i$) interact through the transmission line (G, B) and power load P_i
- Stability of these equations determines how loads on the grid are accommodated

State: rotor angles, velocities ($\delta_i, \dot{\delta}_i$)

Inputs: power loading on the grid (P_i)

Outputs: voltage levels and frequency (based on rotor speed)

Parameters: additional constants required to describe dynamics (B, G, ω_0)

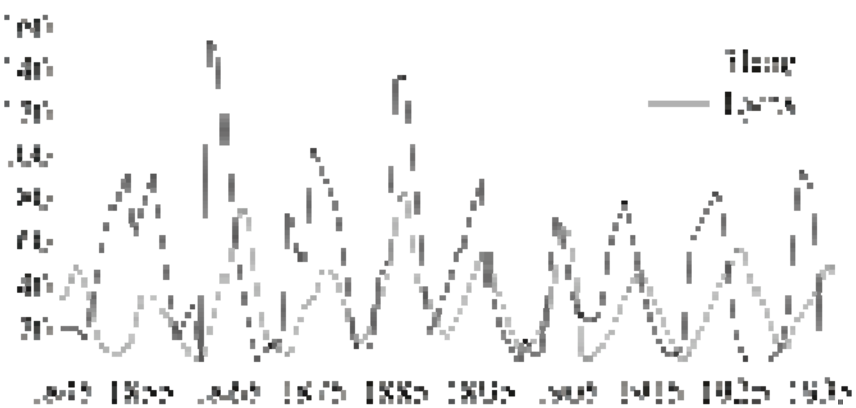
Difference Equations

Difference equations model discrete transitions between continuous variables

- "Discrete time" description (clocked transitions)
- New state is function of current state + inputs
- State is represented as a *continuous* variable

$$x[k+1] = f(x[k], u[k])$$
$$y[k] = h(x[k])$$

Example: predator prey dynamics



Questions we want to answer

- Given the current population of hares and lynxes, what will it be next year?
- If we hunt down lots of lynx in a given year, how will the populations be affected?
- How do long term changes in the amount of food available affect the populations?

Modeling assumptions

- Track population annually (discrete time)
- The predator species is totally dependent on the prey species as its only food supply
- The prey species has an external food supply and no threat to its growth other than the predator.

Example #2: Predator Prey Modeling

Discrete Lotka-Volterra model

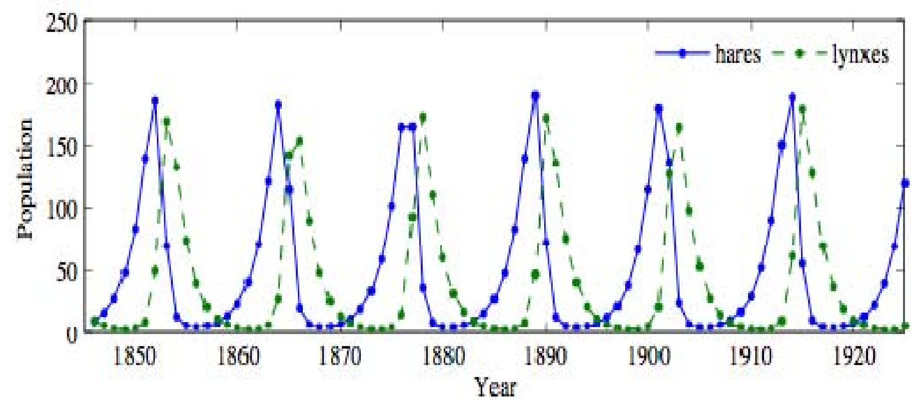
- State
  - $H[k]$ # of rabbits in period k
  - $L[k]$ # of foxes in period k
- Inputs (optional)
  - $u[k]$ amount of rabbit food
- Outputs: # of rabbits and foxes
- Dynamics: Lotka-Volterra eqs

$$H[k+1] = H[k] + b_r(u)H[k] - aL[k]H[k],$$
$$L[k+1] = L[k] + cL[k]H[k] - d_f L[k],$$

- Parameters/functions
  - $b_r(u)$ hare birth rate (per period); depends on food supply
  - d_f lynx mortality rate (per period)
  - a, c interaction terms

MATLAB simulation

- Discrete time model, “simulated” through repeated addition



Comparison with data