ME/CS 133(a): Solution to Homework #2

Problem 1: (Problem 4(a,b) in Chapter 2 of MLS).

Part (a): Let's assume that the statement in part (b) of the problem is true. Let $\vec{w}$ be a 3×1 vector and let $\vec{v}$ be any 3×1 vector. Then:

$$\begin{aligned}(R\hat{w}R^T)\vec{v} &= R\hat{w}(R^T\vec{v}) \\ &= R(\vec{w} \times (R^T\vec{v})) \\ &= (R\vec{w}) \times (RR^T\vec{v}) \\ &= (R\vec{w}) \times \vec{v} \\ &= \widehat{(R\vec{w})}\vec{v}\end{aligned}$$

Since this must be true for any vector $\vec{v}$, then $R\hat{w}R^T = (R\vec{w})\hat{}$.

Part (b): We can now assume that part (a) holds.

$$\begin{aligned}(R\vec{v}) \times (R\vec{w}) &= \widehat{(R\vec{v})}(R\vec{w}) \\ &= (R\hat{v}R^T)(R\vec{w}) \\ &= R\hat{v}R^TR\vec{w} \\ &= R(\hat{v}\vec{w}) \\ &= R(\vec{v} \times \vec{w})\end{aligned}$$

Proving (b) without using (a):

Note that the proof above shows only that statements a) and b) are equivalent, but not to what extent they hold. Though this was considered sufficient for the homework assignment, we here offer a proof of part b) that does not rely on a).

We would like to show that $R(\vec{v} \times \vec{w}) = (R\vec{v}) \times (R\vec{w})$. To do so, we use the fact that for any two vectors $\vec{a}$ and $\vec{b} \in \mathbf{R}^3$, $\vec{a} \times \vec{b} = ||\vec{a}|| * ||\vec{b}|| \sin\theta * \vec{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\vec{n}$ is the unit vector normal to both $\vec{a}$ and $\vec{b}$ in the direction given by the right-hand rule.

Next, we define some notation. Let θ_1 be the angle between $\vec{v}$ and $\vec{w}$, and $\vec{n}_1$ be the unit normal vector orthogonal to $\vec{v}$ and $\vec{w}$. Let θ_2 be the angle between $R\vec{v}$ and $R\vec{w}$, and $\vec{n}_2$ be the unit normal vector corresponding to $R\vec{v}$ and $R\vec{w}$. In both cases, the unit normal vectors point in the right-handed direction.

Simplifying the left-hand side: $R(\vec{v} \times \vec{w}) = R(||\vec{v}|| * ||\vec{w}|| \sin\theta_1 * \vec{n}_1) = ||\vec{v}|| * ||\vec{w}|| \sin\theta_1 * (R\vec{n}_1)$.

Simplifying the right-hand side: $(R\vec{v}) \times (R\vec{w}) = ||\vec{Rv}|| * ||\vec{Rw}|| \sin\theta_2 * \vec{n}_2 = ||\vec{v}|| * ||\vec{w}|| \sin\theta_2 * \vec{n}_2$, where the last equality comes from the fact that R is a rotation matrix and therefore preserves vector lengths.

Setting these expressions for the left and right sides equal, we see that the statement we're trying to prove is equivalent to $||\vec{v}|| * ||\vec{w}|| \sin\theta_1 * (R\vec{n}_1) = ||\vec{v}|| * ||\vec{w}|| \sin\theta_2 * \vec{n}_2$. We divide by $||\vec{v}||$ and $||\vec{w}||$ (we can assume that these are nonzero, as if either were zero, equality would hold trivially).

Thus, it suffices to prove that:

$$\sin\theta_1 * (R\vec{n}_1) = \sin\theta_2 * \vec{n}_2. \tag{1}$$

We argue that $R\vec{n}_1$ is a unit vector and is orthogonal to both $R\vec{v}$ and $R\vec{w}$. We have that $||R\vec{n}_1|| = ||\vec{n}_1|| = 1$, and $(R\vec{n}_1)^T(R\vec{v}) = \vec{n}_1^T R^T R\vec{v} = \vec{n}_1^T\vec{v} = 0$ (since $\vec{n}_1$ is perpendicular to $\vec{v}$ by definition), and similarly for $R\vec{w}$. This proves that $R\vec{n}_1 = \pm\vec{n}_2$.

Next, we show that $\cos\theta_1 = \cos\theta_2$. By definition of the cosine, we know that $\vec{a} \cdot \vec{b} = \vec{a}^T\vec{b} = ||\vec{a}|| * ||\vec{b}|| \cos\theta$, with $\vec{a}$, $\vec{b}$, and θ all defined as before. Using properties of orthogonal matrices, we can see that:

$$\cos\theta_1 = \frac{\vec{v}^T\vec{w}}{||\vec{v}|| * ||\vec{w}||} = \frac{\vec{v}^T(R^TR)\vec{w}}{||R\vec{v}|| * ||R\vec{w}||} = \frac{(R\vec{v})^T(R\vec{w})}{||R\vec{v}|| * ||R\vec{w}||} = \cos\theta_2.$$

This yields two alternatives: either $\theta_1 = \theta_2$ or $\theta_1 = -\theta_2$. We consider these two cases separately.

Case 1: $\theta_1 = \theta_2$, and so $\sin\theta_1 = \sin\theta_2$. We know from earlier that $R\vec{n}_1$ is a unit normal vector to $(R\vec{v})$ and $(R\vec{w})$. Since the angle between $\vec{v}$ and $\vec{w}$ does not change signs under the rotation, $R\vec{n}_1$ will be in the right-handed orientation. So, $\vec{n}_2 = R\vec{n}_1$ and therefore $\sin\theta_1 * (R\vec{n}_1) = \sin\theta_2 * \vec{n}_2$.

Case 2: $\theta_1 = -\theta_2$, and so $\sin\theta_1 = -\sin\theta_2$. With a change in the sign of the angle between θ_1 and θ_2, the direction of the normal vector $\vec{n}_2$ would also undergo a sign change (given by the right-hand rule). So, $\vec{n}_2 = -R\vec{n}_1$ in this case. Therefore, we again find that $\sin\theta_1 * (R\vec{n}_1) = \sin\theta_2 * \vec{n}_2$.

Thus, we have proven that (1) holds, which completes the proof that $R(\vec{v} \times \vec{w}) = (R\vec{v}) \times (R\vec{w})$.

Problem 2: (Problem 3(c) of chapter 2 in the MLS text).

Let rotation matrix R take the symbolic form:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{2}$$

The solution results from an expansion of the determinant of the matrix R (along the first column):

$$\det(R) = r_{11}(r_{22}r_{33} - r_{32}r_{23}) + r_{21}(r_{32}r_{13} - r_{12}r_{33}) + r_{31}(r_{12}r_{23} - r_{22}r_{13}).$$

Note that if the second and third columns of R are denoted

$$\vec{r}_2 = \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix} \qquad \vec{r}_3 = \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix},$$

then $\vec{r}_2 \times \vec{r}_3$ takes the symbolic form:

$$\vec{r}_2 \times \vec{r}_3 = \begin{bmatrix} r_{22}r_{33} - r_{32}r_{23} \\ r_{32}r_{13} - r_{12}r_{33} \\ r_{12}r_{23} - r_{22}r_{13} \end{bmatrix}.$$

Hence, we can see that $\det(R) = \vec{r}_1 \cdot (\vec{r}_2 \times \vec{r}_3)$.

This expression is true for any 3×3 matrix. Note that if $R \in SO(3)$, then $\vec{r}_2$ can always be identified with the "y-axis" of a reference frame, while $\vec{r}_3$ is associated with a "z-axis." Hence, $\vec{r}_2 \times \vec{r}_3$ is associated with the "x-axis," whose dot product with $\vec{r}_1$ (which is the associated "x-axis") must have value 1.

Problem 3: (Problem 8(b,c) in Chapter 2 of the MLS text).

Solution to 8(b):

Firstly, note that $(g\Lambda g^{-1})^n = g(\Lambda^n)g^{-1}$. We prove this by induction. The base case, $n = 1$, is clear. If we assume that $(g\Lambda g^{-1})^n = g(\Lambda^n)g^{-1}$ for some n, then $(g\Lambda g^{-1})^{n+1} = (g\Lambda g^{-1})^n * (g\Lambda g^{-1}) = g(\Lambda^n) * (g^{-1}g) * \Lambda g^{-1} = g(\Lambda^n) * I * \Lambda g^{-1} = g\Lambda^{n+1}g^{-1}$, thus proving our statement.

$$\begin{aligned} e^{g\Lambda g^{-1}} &= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda g^{-1})^2 + \frac{1}{3!}(g\Lambda g^{-1})^3 + \cdots \\ &= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda^2 g^{-1}) + \frac{1}{3!}(g\Lambda^3 g^{-1}) + \cdots \\ &= g(I + \frac{1}{1!}\Lambda + \frac{1}{2!}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \cdots)g^{-1} \\ &= ge^{\Lambda}g^{-1} \end{aligned}$$

Solution to 8(c): Assuming that Λ is constant and θ is a function of time:

$$\begin{aligned} \frac{d}{dt}e^{\Lambda\theta} &= \frac{d}{dt}(I + \frac{1}{1!}\theta\Lambda + \frac{1}{2!}\theta^2\Lambda^2 + \cdots) \\ &= \frac{1}{1!}\dot{\theta}\Lambda + \frac{1}{2!}2\dot{\theta}\theta\Lambda^2 + \cdots \\ &= \dot{\theta}\Lambda(I + \frac{1}{1!}\theta\Lambda + \cdots) = \dot{\theta}\Lambda e^{\Lambda\theta} \end{aligned}$$

Also, since θ is a scalar and Λ commutes with itself, we can equivalently write:

$$\begin{aligned}\frac{d}{dt}e^{\Lambda\theta} &= \frac{1}{1!}\dot{\theta}\Lambda + \frac{1}{2!}2\dot{\theta}\theta\Lambda^2 + \cdots \\ &= (I + \frac{1}{1!}\theta\Lambda + \cdots)\dot{\theta}\Lambda = e^{\Lambda\theta}\dot{\theta}\Lambda\end{aligned}$$

Problem 4: Let Z-Y-X Euler angles be denoted by ψ, ϕ, and γ.

- **Part (a):** Develop an expression for the rotation matrix that describes the Z-Y-X rotation as a function of the angles ψ, ϕ, and γ.

  Rotation about the z-axis by angle ψ can be represented by a rotation matrix whose form can be determined from the Rodriguez Equation:

$$Rot(\vec{z},\psi) = I + \sin\psi\hat{z} + (1-\cos\psi)\hat{z}^2 = \begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}.$$

  Using the Rodriguez equation, the rotations about the y-axis and x-axis can be similarly found as:

$$Rot(\vec{y},\phi) = \begin{bmatrix}\cos\phi & 0 & \sin\phi\\ 0 & 1 & 0\\ -\sin\phi & 0 & \cos\phi\end{bmatrix} \qquad Rot(\vec{x},\gamma) = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\gamma & -\sin\gamma\\ 0 & \sin\gamma & \cos\gamma\end{bmatrix}.$$

  Multiplying the matrices yields the result:

$$\begin{aligned}R(\psi,\phi,\gamma) &= Rot(\vec{z},\psi)\ Rot(\vec{y},\phi)\ Rot(\vec{x},\gamma)\\ &= \begin{bmatrix}c\psi\ c\phi & (c\psi\ s\phi\ s\gamma\ - s\psi\ c\gamma) & (c\psi\ s\phi\ c\gamma\ + s\psi\ s\gamma)\\ s\psi\ c\phi & (s\psi\ s\phi\ s\gamma\ + c\psi\ c\gamma) & (s\psi\ s\phi\ c\gamma\ -\ c\psi\ s\gamma)\\ -s\phi & c\phi\ s\gamma & c\phi\ c\gamma\end{bmatrix}\end{aligned} \tag{3}$$

  where $c\phi$ and $s\phi$ are respectively shorthand notation for $\cos\phi$ and $\sin\phi$, etc.

- **Part (b):** Given a rotation matrix of the form:

$$R = \begin{bmatrix}r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix} \tag{4}$$

  compute the angles ψ, ϕ, and γ as a function of the r_{ij}.

  Direct observation of the matrices in Equations (3) and (4) show that:

$$-\sin\phi = r_{31}\ .$$

Because $\sin(\pi - \phi) = \sin\phi$, there are two solutions to this equation: $\phi_1 = -\sin^{-1}(r_{31})$, and $\phi_2 = \pi - \phi_1$. Similar matchings of the matrix components yield (assuming $\cos\phi \neq 0$):

$$a_{11} = c\psi c\phi, a_{21} = s\psi c\phi \rightarrow \psi = Atan2[\frac{r_{21}}{\cos\phi}, \frac{r_{11}}{\cos\phi}]$$
$$a_{32} = c\phi s\gamma, a_{33} = c\phi c\gamma \rightarrow \gamma = Atan2[\frac{r_{32}}{\cos\phi}, \frac{r_{33}}{\cos\phi}],$$

where the value ϕ_1 or ϕ_2 is used consistently. Thus, there are two equivalent triples (ψ, ϕ, γ) that result in rotation matrix R.

Problem 5: (5 points, Problem 10(b) in Chapter 2 of the MLS text).

Note that

$$\hat{\omega} = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} = wJ \quad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then:

$$\hat{\omega}^2 = w^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -w^2 I; \quad \hat{\omega}^3 = -w^3 J$$

Hence the exponential of $\hat{\omega}$ can be computed as:

$$\begin{aligned} \exp(\theta\hat{\omega}) &= \left(I + \frac{\theta}{1!}\hat{\omega} + + \frac{\theta^2}{2!}\hat{\omega}^2 + \cdots \right) \\ &= \left(I + \frac{w\theta}{1!}J - \frac{w^2\theta^2}{2!}I - \frac{w^3\theta^3}{3!}J + \cdots \right) \\ &= \left(1 - \frac{w^2\theta^2}{2!} + \cdots \right) I + \left(\frac{w\theta}{1!} - \frac{w^3\theta^3}{3!} + \cdots \right) J \\ &= \begin{bmatrix} \cos(w\theta) & -\sin(w\theta) \\ \sin(w\theta) & \cos(w\theta) \end{bmatrix}, \end{aligned}$$

where we utilize the Taylor expansions of the sine and cosine functions.

While you weren't asked to consider this part of the problem, note that the exponential map from $so(2)$ to $SO(2)$ is surjective, as every point in $SO(2)$ can be covered by a point in $so(2)$. This map is not injective since $\exp(\theta\hat{\omega}) = \exp((\theta + 2\pi)\hat{\omega})$.