CDS 101/110 Homework #4 Solution

Problem 1 (CDS 101, CDS 110): (35 points)

(a) Let $z = (x, y, \dot{x}, \dot{y})$. Rewrite the system as

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 9w^2 & 0 & 0 & 2w \\ 0 & -4w^2 & -2w & 0 \end{bmatrix} z$$

Find the eigenvalue of the matrix which are $-0.3465 \pm 0.5548i$ and $0.3465 \pm 0.5548i$. There is an eigenvalue with a positive real part. Therefore, the equilibrium point is unstable.

(b)

$$k_1 = 2.0138 \quad k_2 = 0.6933 \quad k_3 = 2.7039 \quad k_4 = 2.8166$$

Code:

```
w = 2*pi/29;
A = [0 1 0 0 ;
    0 0 0 1;
    9*w^2 0 0 2*w;
    0 -4*w^2 -2*w 0];
B = [0 0 0 1]';

eig(A)

place(A, B, [-3*w -4*w -3*w+3*w*1i -3*w-3*w*1i])
```

(c) (This problem will not be graded because the phase plot (x, y) can be any arbitrary phase plot.)

Problem 2 (CDS 101, CDS 110): (5 points)

Set $z(t) = x(t) - e^{At}x(0)$. Then, $z(t)$ can reach any states from zero initial state according to the argument in the texbook on page 7-3. When $z(t)$ can reach any state, we can represent $z(t)$ by some basis vectors (i.e. W_r) that span the entire state space. We can also write $x(0)$ using the same set of basis vectors (i.e. W_r). So, $x(t) = e^{At}x(0) + z(t)$ can be written using the basis vectors in W_r as well. Hence, $x(t)$ is reachable from a non-zero initial state.

Problem 3 (CDS 110): (25 points)

Let $z = (\phi, \delta, \dot{\phi}, \dot{\delta})$.

Feedback gain K:

$$k_1 = 0.3247 \quad k_2 = 8.4043 \quad k_3 = -1.3455 \quad k_4 = 0.1071$$

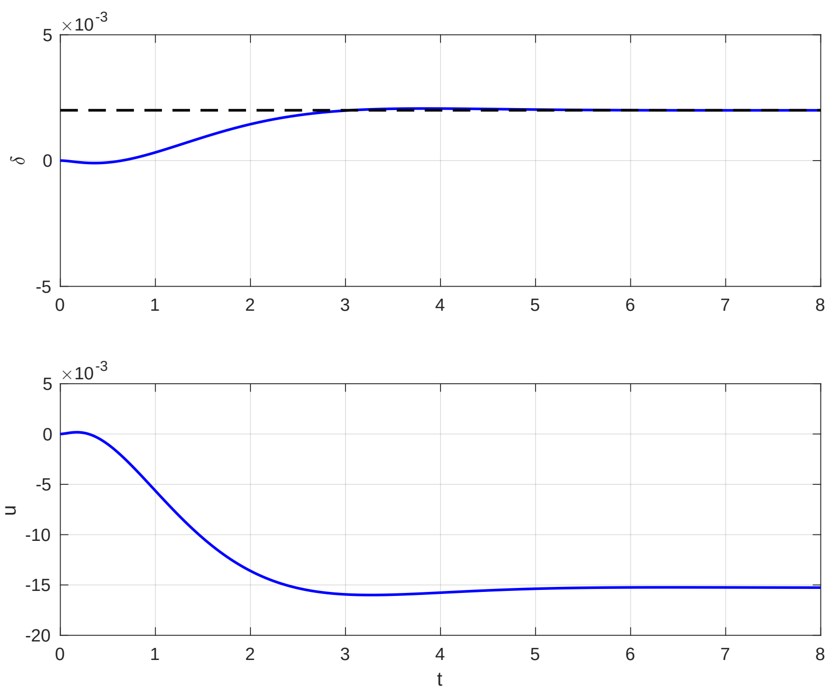
Reference gain:

$$K_r = -0.5429$$

Code:

```
run('Bicycle_whipple.m')

% Desired closed loop eigenvalues
P=[1*[-1+1i -1-1i] -2 -10];
K=place(A,B,P);
Acl=A-B*K;
C=[0 1 0 0];
Kr=-1/(C/Acl*B);
disp('Feedback gain K=');disp(K)
disp('Refererence gain Kr=');disp(Kr)

% Simulate
bike_cls=ss(Acl,B*Kr,C,0);
t=0:0.01:8;

[y,t,x]=step(bike_cls,t);

u=Kr*0.002-K*x';

subplot(211);
pl=plot(t,x(:,2)/500,'b-',t,0.002*ones(1,length(t)),'k--');
set(pl,'Linewidth',1.5);grid on;
axis([0 8 -0.005 0.005]);
ylabel('$\delta$','Interpreter','latex');
subplot(212);
pl=plot(t,u/500,'b-');
set(pl,'Linewidth',1.5);grid on;
xlabel('t');ylabel('u');
```

Plot: The dashed line is the reference at 0.002.



Problem 4 (CDS 110): (15 points)

Let λ_k be an eigenvalue of A. Then,

$$\lambda(\lambda_k) = \lambda_k^n + a_1\lambda_k^{n-1} + \cdots + a_{n-1}\lambda_k + a_n = 0.$$

Let Λ be a diagonal matrix that has the eigenvalues of A at its diagonal. Then,

$$\Lambda^n + a_1\Lambda^{n-1} + \cdots + a_{n-1}\Lambda + a_n I = 0.$$

Because A is diagonalizable, $A = T\Lambda T^{-1}$. Furthermore, $A^k = T\Lambda^k T^{-1}$. So,

$$\begin{aligned}\lambda(A) &= A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I \\ &= T\Lambda^n T^{-1} + a_1 T\Lambda^{n-1}T^{-1} + \cdots + a_{n-1}T\Lambda T^{-1} + a_n I \\ &= T(\Lambda^n + a_1\Lambda^{n-1} + \cdots + a_{n-1}\Lambda + a_n I)T^{-1} \\ &= 0\end{aligned}$$