Problem Set 3

Problem 1

Consider the system whose dynamics are given by:

$$\tau \frac{dx}{dt} = -x + u \qquad y = x$$

We know the response is $y(t) = CA^{-1}e^{At}B + D - CA^{-1}B$ where $C = 1,\ B = \frac{1}{\tau},\ A = -\frac{1}{\tau}$.
Plugging in and simplifying, we get $y(t) = 1 - e^{\frac{t}{\tau}}$
Time to get to $0.1y_{ss} \quad \rightarrow \quad 0.1 = 1 - e^{\frac{t}{\tau}} \quad \rightarrow \quad t = 0.105\tau$
Time to get to $0.9y_{ss} \quad \rightarrow \quad 0.9 = 1 - e^{\frac{t}{\tau}} \quad \rightarrow \quad t = 2.3\tau$.
Rise time $t_r = (2.3 - 0.105)\tau = 2.2\tau \approx 2\tau$.

1% Settling time: $\quad \rightarrow \quad 0.99 = 1 - e^{\frac{t}{\tau}} \quad \rightarrow \quad t = 4.6\tau$

2% Settling time: $\quad \rightarrow \quad 0.98 = 1 - e^{\frac{t}{\tau}} \quad \rightarrow \quad t = 3.91\tau \approx 4\tau$

5% Settling time: $\quad \rightarrow \quad 0.95 = 1 - e^{\frac{t}{\tau}} \quad \rightarrow \quad t = 3.0\tau$

Problem 2

Consider the system

$$\ddot{x} + 2\zeta w_0 \dot{x} + w_0^2 x = w_0^2 u$$

Part (a): Convert the dynamic system to first order form
Denote $x_1 = x, \quad x_2 = \dot{x}$.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w_0^2 & -2\zeta w_0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ w_0^2 \end{bmatrix} u$$

Part (b): Determine and plot the impulse response of this system for the case C = [1 0]

Response $h(t) = Ce^{At}B$

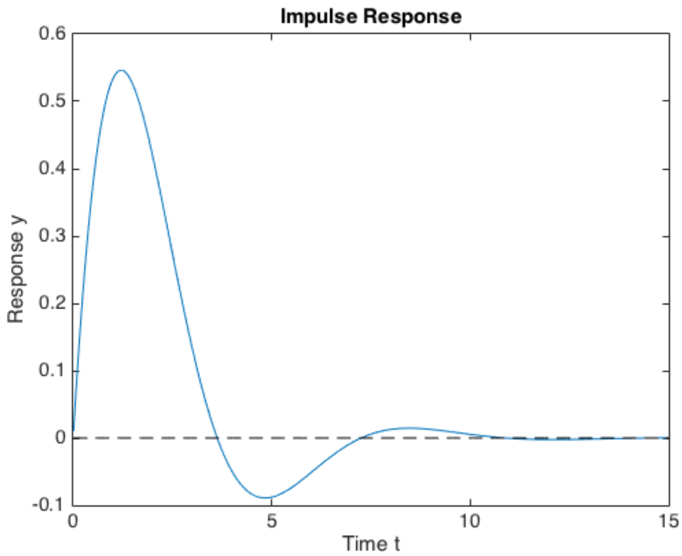
$A = VDV^{-1}$ where

$$V = \begin{bmatrix} -\frac{(\zeta+\sqrt{\zeta^2-1})}{w_0} & -\frac{(\zeta-\sqrt{\zeta^2-1})}{w_0} \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -w_0(\zeta-\sqrt{\zeta^2-1}) & 0 \\ 0 & -w_0(\zeta+\sqrt{\zeta^2-1}) \end{bmatrix}$$

$h(t) = Ce^{At}B = CVe^{Dt}V^{-1}B$

$$h(t) = \begin{bmatrix} -\frac{(\zeta+\sqrt{\zeta^2-1})}{w_0} & -\frac{(\zeta-\sqrt{\zeta^2-1})}{w_0} \end{bmatrix} \begin{bmatrix} exp(-w_0(\zeta-\sqrt{\zeta^2-1})) & 0 \\ 0 & exp(-w_0(\zeta+\sqrt{\zeta^2-1})) \end{bmatrix} \begin{bmatrix} -\frac{w_0^2(\zeta-\sqrt{\zeta^2-1})}{2\sqrt{\zeta^2-1}} \\ \frac{w_0^2(\zeta+\sqrt{\zeta^2-1})}{2\sqrt{\zeta^2-1}} \end{bmatrix}$$

The impulse response for $w_0 = 1$, $\zeta = 0.5$ is shown below (with code for finding the impulse response and generating the plot).



```
%Find Impulse Response
syms w0 zed t
A = [0 1; -w0^2 -2*zed*w0];
B = [0; w0^2];
C = [1 0];
[V, D] = eig(A);
%Get the eigendecomposition
V = simplify(V);
D = simplify(D);
y_unit = simplify(C*A^-1*V*expm(D*t)*V^-1*B - C*A^-1*B);

%Pick w_0=1 and zed=0.5 and plot impulse response
```

```
w0 = 1;
zed = 0.5;
A = [0 1; -w0^2 -2*zed*w0];
B = [0; w0^2];
C = [1 0];

time = 1500;
t = zeros(1,time);
response = zeros(1,time);
hline = zeros(1,time);
for i = 1:time
    t(i) = i*0.01;
    response(i) = C*expm(A*t(i))*B;
end
plot(t, response)
hold on
plot(t, hline, 'k--')
hold off
```

Part (c): Find the response of this system to a unit step input, assuming that $x(0) = 0$, $\dot{x}(0) = 0$.

Response $h(t) = CA^{-1}e^{At}B - CA^{-1}B$

$$h(t) = 1 - e^{-\zeta w_0 t} cos(w_d t) - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta w_0 t} sin(w_d t)$$

where $w_d = w_0\sqrt{\zeta^2 - 1}$.
See Lectures notes from 10/21/2016 (Slides 5-8) for detailed derivation.

Part (d): Determine the time until the first peak in response. Knowing this time, derive an expression for the peak overshoot.

$$t_{peak} = \frac{\pi}{w_0\sqrt{1-\zeta^2}}$$

$$y_{peak} = 1 - \frac{exp(\pi\zeta/\sqrt{1-\zeta^2})}{\sqrt{1-\zeta^2}} sin(\pi + \theta)$$

where $\theta = cos^{-1}(\zeta)$.
See Lectures notes from 10/21/2016 (Slides 5-8) for detailed derivation.

Part (e): Estimate the rise time, which is the time it takes from the onset of the step input until the time that the response first reaches a magnitude of one (the amplitude of the step input).
The step response can be written:

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_0 t} sin(w_d t + \zeta)$$

When $y = 1$, then $\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_0 t} sin(w_d t + \theta) = 0$
Thus $sin(w_d t + \theta) = sin(w_0 \sqrt{\zeta^2 - 1} t + cos^{-1}(\zeta)) = 0$

$$t_r = \frac{\pi - cos^{-1}(\zeta)}{w_0 \sqrt{\zeta^2 - 1}}$$

Problem 3

Show that $P = \int_0^\infty e^{A^T \tau} Q e^{A\tau} d\tau$ defines a Lyapunov function of the form $V = x^T P x$ given that Q is positive definite.
For fixed x, $V = x^T P x = \int_0^\infty (e^{A\tau} x)^T Q (e^{A\tau} x) d\tau$.
Since Q is positive definite, then $(e^{A\tau} x)^T Q (e^{A\tau} x) > 0 \ \ \forall x / \{0\} \in \mathbb{R}^n, \tau \in \mathbb{R}$
Therefore V(x) > 0 and V(0) = 0.

$$\dot{V} = \dot{x^T} P x + x^T P \dot{x} = x^T (A^T P + PA) x$$

$$A^T P + PA = \int_0^\infty (A^T e^{A^T \tau} x Q e^{A\tau} + e^{A^T \tau} x Q e^{A\tau} A) d\tau = \int_0^\infty \frac{d}{d\tau} (e^{A^T \tau} x Q e^{A\tau}) d\tau = -Q$$

Therefore

$$\dot{V} = x^T (A^T P + PA) x = -x^T Q x$$

Since Q is positive definite, then $\dot{V} < 0$ and we can conclude that $V = x^T P x$ defines a valid Lyapunov function.

Problem 4

Part (a):

$$x[k+1] = Ax[k] + Bu[k] \qquad\qquad y[k] = Cx[k] + Du[k]$$

Proof by induction.
Consider

$$x[k] = A^k x[0] + \sum_{i=0}^{k-1} A^{k-1-i} B u[i]$$

At the initial point,

$$x[1] = Ax[0] + Bu[0]$$

Now,

$$\begin{aligned} x[k+1] &= A^{k+1}x[0] + \sum_{i=0}^{k} A^{k-i}Bu[i] \\ &= A(A^k x[0]) + \sum_{i=0}^{k-1} A^{k-i}Bu[i] + Bu[k] \\ &= A(A^k x[0]) + A\sum_{i=0}^{k-1} A^{k-1-i}Bu[i] + Bu[k] \\ &= Ax[k] + Bu[k] \end{aligned}$$

So,

$$x[k] = A^k x[0] + \sum_{i=0}^{k-1} A^{k-1-i}Bu[i]$$

Then,

$$y[k] = Cx[k] + Du[k] = CA^k x[0] + \sum_{i=0}^{k-1} CA^{k-1-i}Bu[i] + Du[k]$$

Part (b):
For checking asymptotic stability, assume $u = 0$.
Then $x[k] = A^k x[0]$.
Consider the eigendecomposition of $A = VDV^{-1}$ where D is diagonal with eigenvalues on the diagonal and V is an orthonormal matrix (represents a change of basis).

$$x[k] = A^k x[0] = VD^kV^{-1}x[0]$$

(if direction)
If eigenvalue of A has magnitude strictly less than 1, $D^k \to 0$ as $k \to \infty$ because D is a diagonal matrix and its diagonal are the eigenvalues. Consequently, $x[k] \to 0$. Therefore, the system is asymptotically stable.

(only if direction)
If eigenvalue of A has magnitude equal to 1, $D^k \to D = I$ as $k \to \infty$. Consequently, $x[k] = x[0]$ for all $x[0] \neq 0$.

If eigenvalue of A has magnitude greater than 1, $D^k \to \infty$ as $k \to \infty$. Consequently, $x[k] \to \infty$ for all $x[0] \neq 0$.

Therefore, the systems are not asymptotically stable.

Part (c):
Consider the input $u = e^{iwk}$

$$y[k] = CA^k x[0] + \sum_{j=0}^{k-1} CA^{k-j-1}Be^{iwj} + De^{iwk}$$

If we assume asymptotic stability, then $CA^k x[0] \to 0$.

$$y[k] = C\sum_{j=0}^{k-1} A^{k-j-1}Be^{iwj} + De^{iwk} = C\sum_{j=0}^{k-1} \frac{A^{k-j-1}}{e^{iw(k-j-1)}}(e^{iw(k-1)}) + De^{iwk}$$

$$y[k] = C(I - \frac{A}{e^{iw}})^{-1}(e^{iw})^{-1}e^{iwk} + De^{iwk} = [C(e^{iw}I - A)^{-1} + D]e^{iwk}$$

Thus the response to e^{iwk} is:

$$y[k] = [C(e^{iw}I - A)^{-1} + D]e^{iwk}$$

Thus by linearity, the response to $sin(wk) = \frac{e^{iwk} - e^{-iwk}}{2i}$ will be:

$$y[k] = \frac{[C(e^{iw}I - A)^{-1} + D]e^{iwk} - [C(e^{-iw}I - A)^{-1} + D]e^{-iwk}}{2i}$$

Part (d):
As in the continuous time case, we let $z = x - x_e$, $v = u - u_e$, and $w = y - h(x_e, u_e)$. Expanding the dynamics in a Taylor series, we have:

$$x[k+1] = f(x_e, u_e) + \frac{df}{dx}(x[k] - x_e) + \frac{df}{du}(u[k] - u_e) + h.o.t.$$

$$y[k] = h(x_e, u_e) + \frac{dh}{dx}(x[k] - x_e) + \frac{dh}{du}(u[k] - u_e) + h.o.t.$$

The resulting linearized system is obtained by assuming the higher order terms can be neglected and the dynamics become:

$$z[k+1] = Az[k] + Bv[k] \qquad w[k] = Cz[k] + Dv[k]$$

where

$$A = \left.\frac{df}{dx}\right|_{(x_e, u_e)} \qquad B = \left.\frac{df}{du}\right|_{(x_e, u_e)} \qquad C = \left.\frac{dh}{dx}\right|_{(x_e, u_e)} \qquad D = \left.\frac{dh}{du}\right|_{(x_e, u_e)}$$