CDS 101/110: Lecture 5.2
Observability & State Estimation

October 26, 2016

Goals:

- Introduce notion of Observability
- Observability Test and Observable Canonical Form.
- Observer Design
- State feedback of estimated state

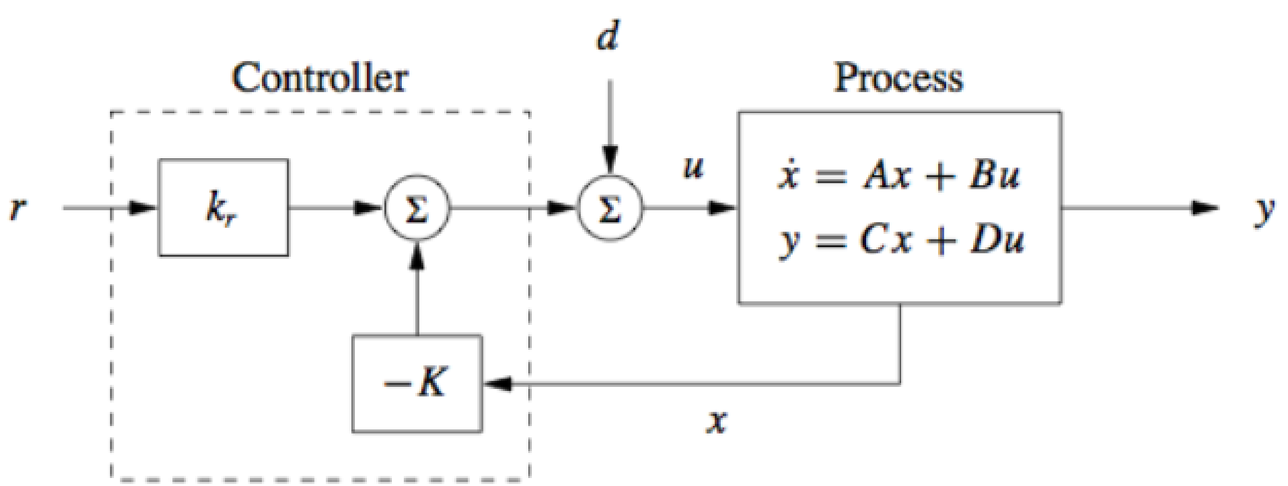
Reading:

- Åström and Murray, Feedback Systems-2e, Section 8.1-8.2, and first half of 8.3

Types of Feedback

State Feedback: $u(t) = -Kx(t)$

- Can place poles arbitrarily if system is reachable
- Can relate poles to performance criteria, such as overshoot.
- Can add "dynamic compensator", such as integral feedback, which overcomes modeling errors or uncertainty.
- *But,* states cannot always be measured, as needed for feedback.



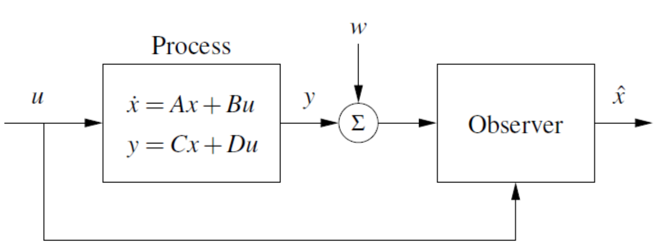
Types of Feedback

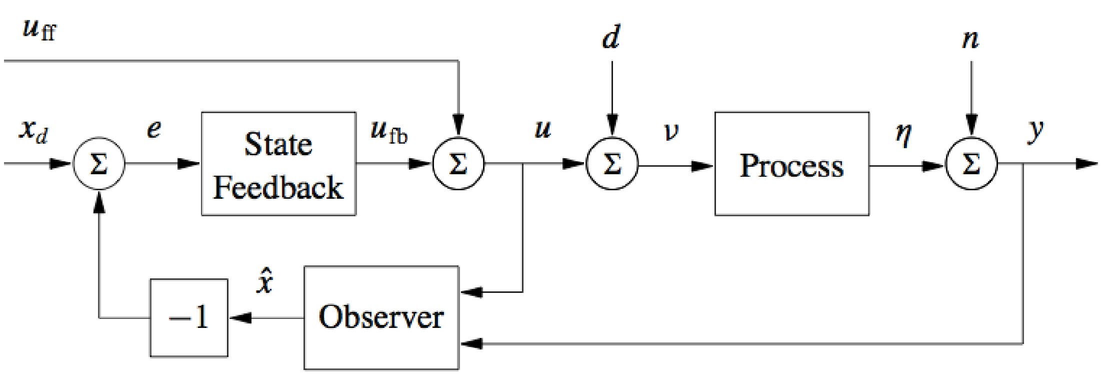
Output Feedback:

- Can we stabilize the system using only the output measurements, $y(t)$?
- ***Approach #1***: develop a theory which directly addresses this problem
- ***Approach #2:*** connect to state feedback theory by trying to determine if states is can be “reconstructed”, “inferred”, or “estimated” from $y(t)$.

Key Questions:

- Are internal states, $x(t)$, “observable” from output $y(t)$?
- How do we actually estimate $x(t)$ from $y(t)$?
- Does the use of output feedback limit performance?





Observability

System: $\dot{x} = Ax + Bu; \quad y = Cx + Du$ (*)

- **Definition:** The linear system (*) is said to be ***Observable*** if for every T>0 it is possible to determine the system state $x(T)$ through measurements $y(t)$ and knowledge of $u(t)$ on the interval $[0, T]$.
  - Note: some texts/papers are slightly different: Observable if $x(t = 0)$ can be determined from measurements and inputs.
  - If (*) is observable, then there are no "hidden" internal states. This is a practical issue in system design—do you have the right sensors?

Testing for Observability:

- Simplify the problem by ignoring controls: $\dot{x} = Ax; \quad y = Cx$ (**)
  - We can do this because of linearity!d
  - If C is square and full rank, then solution is easy: $x(T) = C^{-1}y(T)$.
  - But that's generally not the case

Observability *(continued)*

Construct Estimate as follows:

- Take the derivative of $y(t)$: $\dot{y} = C\dot{x} = CAx$
  - Note: some texts/papers are slightly different: Observable if $x(t=0)$ can be determined from measurements and inputs.
  - If (*) is observable, then there are no “hidden” internal states. This is a practical issue in system design—do you have the right sensors?
  - Continue taking derivatives of output: $\ddot{y}(t) = \frac{d}{dt}(CAx) = CA^2x(t)$, …
  - Arrange all of the derivatives in in a column

$$\begin{bmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \\ \vdots \\ W_O \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x(t) \equiv W_O x(t)$$

Observability Matrix

  - Why can we stop at A^{n-1}?

Observability *(continued)*

Aside: Cayley-Hamilton Theorem

- Let A be an $n \times n$ matrix.
- Let $\lambda_A(s) = \det(sI - A) = s^n + a_1\, s^{n-1} + \cdots + a_{n-1}s + a_n$ be characteristic polynomial of A.
- A satisfies its own characteristic polynomial: $A^n + a_1\, A^{n-1} + \cdots + a_{n-1}A + a_n I = 0$
  - Hence, A^k for $k \geq n$ are linear combinations of $I, A, \cdots, A^{n-1}$

Theorem:

- A linear system (**) is ***observable*** if and only if W_O (the observability matrix) is full rank. For a single output system, the is equivalent to W_O begin full rank. In this case: $x(T) = W_O^{-1} \begin{bmatrix} y(T) \\ \vdots \\ \frac{d^{n-1}}{dt^{n-1}} y(T) \end{bmatrix}$